5 Integral Calculus

5.1 The Riemann Integral

Let $S = [a, b]$ be a closed interval, and let f be a function that is bounded on S, i.e. there is a number, C, such that $|f(x)| \leq C$ for all $x \in S$. A **partition** of $[a, b]$ is a set P of points $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

A **refinement** of a partition P is another partition P' such that $P \subset P'$. Define

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}), \text{ where } m_k = \inf \{f(x) \mid x \in [x_{k-1}, x_k]\},$$
$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}), \text{ where } M_k = \sup \{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

Now if P and P' are partitions of S and P' is a refinement of P, then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P). \tag{49}$$

This is easy to see geometrically. In fact, the coarsest partition bounds the set of all $L(f, P)$ and $U(f, P)$ with respect to partitions, since

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a),$$

where $m = \inf \{f(x) \mid x \in S\}$ and $M = \sup \{f(x) \mid x \in S\}$. Now define

$$\mathcal{L} = \sup_P \{L(f, P)\} \text{ and } \mathcal{U} = \inf_P \{U(f, P)\}.$$

It is easy to see that $\mathcal{L} \leq \mathcal{U}$. The argument is as follows. For a fixed partition P^*, $L(f, P^*)$ is a lower bound for the set of $U(f, P)$, because for any two partitions P and P', there is a refinement of both of them, P'', which by (49), leads to

$$L(f, P) \leq L(f, P'') \leq U(f, P'') \leq U(f, P').$$

Therefore, $L(f, P) \leq \mathcal{U}$, and this is true for every P, completing the argument.

A bounded function, $f : S \to \mathbb{R}$, is **Riemann integrable** if $\mathcal{L} = \mathcal{U}$. If this is the case, the common value is denoted $\int_S f(x)dx$ or $\int_a^b f(x)dx$.

Lemma 3. *f is Riemann integrable on $[a, b]$ if and only if for all $\epsilon > 0$ there is a partition P such that*

$$U(f, P) - L(f, P) < \epsilon. \tag{50}$$

Proof. Let us start by assuming that f is Riemann integrable on $[a, b]$ and try to obtain the implication. Let

$$\mathcal{I} = \int_a^b f(x)dx. \tag{51}$$

Take any $\epsilon > 0$. Since $\mathcal{L} = \mathcal{U}$, it follows that there must be partitions P' and P'' such that

$$\mathcal{I} - \frac{\epsilon}{2} < L(f, P') \text{ and } U(f, P'') < \mathcal{I} + \frac{\epsilon}{2},$$

which is just a simple application of (49). From this we get $\mathcal{I} < L(f, P) + \frac{\epsilon}{2}$ and $U(f, P) - \frac{\epsilon}{2} < \mathcal{I}$, which when combined gives us $U(f, P) - L(f, P) < \epsilon$. Check.

We can go the other way now. Assume that (50) is true. By definition, $\mathcal{U} \leq U(f, P)$ for any P and $\mathcal{L} \leq L(f, P)$ for any P. $U(f, P) \geq L(f, P)$ and $\mathcal{U} \geq \mathcal{L}$, so it must be that $0 \leq \mathcal{U} - \mathcal{L} < \epsilon$. But ϵ is arbitrary, so $\mathcal{U} = \mathcal{L}$. Therefore, f is Riemann integrable. □

Exercise 47. We now introduce a new concept. A function $f : [a, b] \to \mathbb{R}$ is **uniformly continuous** if for all $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. This differs from regular continuity in that here δ does not depend on x, whereas in the regular case it may. Show that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.

Theorem 20. *If $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.*

Proof. Now we are going to use the Lemma 3 and Exercise 47 to prove that continuous functions are integrable. Since f is uniformly continuous, by the exercise above it is uniformly continuous; i.e., for any $\epsilon > 0$ there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b - a)$, where x and y are elements of $[a, b]$. (It doesn't matter that we divided ϵ by $b - a$; if you don't like that, you could have chosen $(b - a)\epsilon$ instead; it was arbitrary after all.) Now for any ϵ take the associated δ and partition the interval $[a, b]$ into n subintervals, each of size $l < \delta$. By definition, the m_k and M_k differ by at most $\epsilon/(b - a)$. Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &< \sum_{k=1}^{n} \frac{\epsilon}{b - a} l = \frac{\epsilon}{b - a} \sum_{k=1}^{n} l = \epsilon. \end{aligned}$$ □

Now one thing that I'm not going to prove but that you should know is that if a bounded function is continuous except at a finite number of points, then it is Riemann integrable. There are even stronger results, but we won't need to prove them here.

Before proceeding it will be convenient to state definitions and properties (which we will not prove) relating to the Riemann integral.

Theorem 21. *Let f and g be Riemann integrable on $[a, c]$ and let $k \in \mathbb{R}$. Then,*

1. $\int_a^c kf(x)dx = k\int_a^c f(x)dx$
2. $\int_a^c [f(x) + g(x)]dx = \int_a^c f(x)dx + \int_a^c g(x)dx$
3. *The same as (2) but with minus signs.*
4. $|\int_a^c f(x)dx| \leq \int_a^c |f(x)|dx$
5. $\int_a^c f(x)dx \leq \int_a^c g(x)dx$ *if* $f(x) \leq g(x)$ *for all* $x \in [a, c]$*, and*
6. *If* $b \in [a, c]$*, then* $\int_a^c f(x)dx) = \int_a^b f(x)dx + \int_b^c f(x)dx$

In addition, we also have the following two definitions.

7. $\int_b^b f(x)dx := 0$ *for* $b \in [a, c]$*, and*
8. $\int_a^c f(x)dx := \int_c^a f(x)dx$

5.2 The Fundamental Theorem of Calculus

Now we introduce and prove an important theorem that connects integration with differentiation: the fundamental theorem of calculus. This important result says, roughly, that integration is the inverse operation of differentiation. We first have the following lemma that sets the stage for this result.

Lemma 4. (**Leibniz's Rule**) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous so that it is Riemann integrable. For $x \in [a, b]$ define F by*

$$F(x) = \int_a^x f(t)dt. \tag{52}$$

Then F is differentiable at every point $x \in (a, b)$ and $F'(x) = f(x)$.

Proof. Fix $x \in (a,b)$ and choose $y \in (a,b)$ close to x. Then

$$\frac{F(y)-F(x)}{y-x} = \frac{1}{y-x}\left[\int_a^y f(t)dt - \int_a^x f(t)dt\right] = \frac{1}{y-x}\int_x^y f(t)dt \tag{53}$$

whence

$$\frac{F(y)-F(x)}{y-x} - f(x) = \frac{1}{y-x}\int_x^y [f(t)-f(x)]dt. \tag{54}$$

So given $\epsilon > 0$ we can choose $\delta > 0$ such that $|y-x| < \delta$ implies that the absolute value of the right hand side is less than ϵ. □

Theorem 22. (**Fundamental Theorem of Calculus**) *Suppose that* $f : [a,b] \to \mathbb{R}$ *and* $G : [a,b] \to \mathbb{R}$ *are continuous functions,* G *is differentiable everywhere in* (a,b)*, and* $G'(x) = f(x)$*. Then*

$$\int_a^b f(x)dx = G(b) - G(a)$$

Proof. Take F from the statement of the lemma above, and notice that $F - G$ has derivative 0, so it must be constant. Therefore,

$$F(b) - G(b) = F(a) - G(a) = -G(a) \text{ implies } F(b) = G(b) - G(a).$$

But $F(b) = \int_a^b f(x)dx$, completing the argument. □

5.3 Properties of the Integral

As with the case of the derivative, we now run through a few important properties of the integral that will come handy later on.

Theorem 23. (**integration by parts**) *Suppose that* $f : [a,b] \to \mathbb{R}$ *and* $g : [a,b] \to \mathbb{R}$ *have continuous first derivatives on* (a,b)*. Then if* x *and* y *are elements of* (a,b)*,*

$$\int_x^y f(t)g'(t)dt = \big[f(y)g(y) - f(x)g(x)\big] - \int_x^y f'(t)g(t)dt \tag{55}$$

Proof. Simply integrate the product rule in the form: $fg' = (fg)' - f'(g)$. □

Exercise 48. Find $\int_0^\pi x(\sin x)dx$. (define $\sin x$)

Theorem 24. (**change of variables**) *Suppose that* g *is differentiable on the open interval* (a,b) *and that its derivative is continuous. Let* T *be an open interval such that* $g(x) \in T$ *for all* $x \in (a,b)$*. If a function* f *is continuous on* T *then the composition* $f \circ g$ *is continuous on* (a,b)*, and*

$$\int_a^b [f \circ g](x)g'(x)dx = \int_{g(a)}^{g(b)} f(g)dg \tag{56}$$

Proof. Simply integrate the chain rule. □

Exercise 49. Find $\int_0^1 x\sqrt{(1-x^2)}dx$.

Improper Integrals You will be pleased to know that if f is Riemann integrable on the interval $[a,b]$ its integral on $(a,b]$, $[a,b)$ and (a,b), are all defined to be the same as its integral on $[a,b]$. You will also be pleased to know that $a=-\infty$ and/or $b=\infty$ are allowed so long as you remember that

$$\int_a^\infty f(x)dx \text{ is really } \lim_{b\to\infty}\int_a^b f(x)dx \text{ and } \int_{-\infty}^b f(x)dx \text{ is really } \lim_{a\to-\infty}\int_a^b f(x)dx.$$

5.4 The Integral in Multiple Dimensions

For $X\subset\mathbb{R}^n$ let $f:X\to\mathbb{R}$ be a continuous function. We would like to develop a notion of the integral

$$\int_A f(x)dx$$

that corresponds to computing the volume under f in a region $A\subseteq X$. Suppose, for example, that $A=[a_1,b_1]\times[a_2,b_2]$ so that it is a rectangle in $\mathbb{R}^2$. It seems natural to think of partitioning this rectangle into small rectangles (just as we partitioned the interval in $\mathbb{R}$ into small intervals), compute the volume of f under each small rectangle, and then take the limit as the partition gets finer. This is exactly what we do, even though graphical intuitions may not carry over into higher dimensions.

Suppose that $A=[a_1,b_1]\times\ldots\times[a_n,b_n]$. Let $f^1=f$ and view $f^1(x_1;x_2,\ldots,x_n)$ as a function of only x_1, i.e. we are holding $x_2,\ldots,x_n$ constant. It is good to know that even in the multivariate setting the continuity of f allows us to define the Riemann integral,

$$f^2(x_2;x_3,\ldots,x_n)=\int_{a_1(x_2,\ldots,x_n)}^{b_1(x_2,\ldots,x_n)} f(x_1;x_2,\ldots,x_n)dx_1, \tag{57}$$

which is a continuous function only of $x_2,\ldots,x_n$, but viewed as a function of x_2 while holding $x_3,\ldots,x_n$ constant. Now hold $x_3,\ldots,x_n$ constant and consider

$$f^3(x_3;x_4,\ldots,x_n)=\int_{a_2(x_3,\ldots,x_n)}^{b_2(x_3,\ldots,x_n)} f^2(x_2;x_3,\ldots,x_n)dx_2. \tag{58}$$

Continue this idea until you have integrated with respect to x_n. What you have calculated then is the multiple integral

$$\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(x_1,\ldots,x_n)dx_1\ldots dx_n. \tag{59}$$

Theorem 25. *Suppose that $A = [a_1, b_1] \times ... \times [a_n, b_n] \subseteq X \subset \mathbb{R}^n$ and f is a continuous real-valued function defined on X. Then,*

$$\int_A f(x)dx = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1, \ldots, x_n)dx_1 \ldots dx_n.$$

An implication is that the order of the variables you integrate with respect to does not matter; you get the same answer at the end. This is **Fubini's theorem**.

But what happens if A does not have the form $[a_1, b_1] \times ... \times [a_n, b_n]$? For example, what if A is a circle or a more complicated region, and we want to compute the "volume" under f over the region A. Fortunately, the change of variables theorem generalizes in a way that might help us. You will use it in the next exercise.

Theorem 26. (**generalized change of variables**) *Suppose that f is a continuous real-valued function defined on $\mathbb{R}^n$. Suppose that $U \subset \mathbb{R}^n$ be an open set and $g : U \to \mathbb{R}^n$ an injective function with continuous partial derivatives. Consider a bounded open set $A \subset \mathbb{R}^n$ such that there is a compact set B for which $A \subset B \subset U$ and suppose that the **Jacobian** matrix, $J(x) = [\partial g_i/\partial x_j]_{i=1,\ldots,n}^{j=1,\ldots,n}$, is invertible (i.e. has an inverse) for all $x \in A$. If $[f \circ g](A)$ is bounded, then*

$$\int_{g(A)} f(x)dx = \int_A [f \circ g](x) \left|\det J(x)\right| dx$$

Exercise 50. Let A be the region:

$$A = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -x_1 - 1, x_2 \leq -x_1 + 1, x_2 \geq x_1 - 1, x_2 \leq x_1 + 1\right\}.$$

Now calculate

$$\int_A \left(\frac{x_1 - x_2}{x_1 + x_2 + 2}\right)^2 d(x_1, x_2). \tag{60}$$

Exercise 51. (**differentiating under the integral**) Suppose that $S \subset \mathbb{R}^n$ is an open box and $f : S \times [a, b] \to \mathbb{R}$ is a continuous function with continuous partial derivatives $\partial f/\partial x_i$ for $i = 1, ..., n$. Then the function $\varphi(x) = \int_a^b f(x, t)dt$ is continuously differentiable. Take this as given and prove that

$$\frac{\partial \varphi(x)}{\partial x_i} = \int_a^b \frac{\partial f(x, t)}{\partial x_i} dt.$$

Hint: Simple proofs use Fubini's theorem and Leibniz's rule. If it is more convenient, feel free to focus on the case of $S \subset \mathbb{R}$ being an open interval (i.e. the $n = 1$ case).

5.5 Taylor's Theorem

We now prove two versions of a remarkably useful theorem, the first for functions of one variable and the second for functions of multiple variables. These results have many applications. For example, will use Taylor's theorem in the proof of the important result in probability theory known as the Central Limit Theorem.

Theorem 27. (**Taylor's Theorem in** $\mathbb{R}$) *Let S be an open interval and $f : S \to \mathbb{R}$ a function whose first, second,..., and mth derivatives exist and are all continuous. Denote the jth derivative of f by $f^{(j)}$. For $a, b \in S$, and $n \leq m$ we have*

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + R_n \tag{61}$$

where

$$R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt. \tag{62}$$

Proof. The fundamental theorem of calculus tells us that $f(b) = f(a) + \int_a^b f'(t)dt$. Notice that this is simply equation (61) for $n = 1$. Now assume that (61) is true. Perform the integration in (62) by parts where the function whose derivative is not taken is $f^{(n)}(t)$, and the derived function is $\frac{(b-t)^{n-1}}{(n-1)!}$. If we do this we get

$$R_n = \frac{(b-a)^n}{n!} f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)dt. \tag{63}$$

Plug this back into (61) and, by induction, we are done. □

Exercise 52. Assume in the above theorem that S contains 0 and remember that because $f^{(n)}(x)$ is continuous and $[a, b]$ is compact, the function is bounded on $[a, b]$. Argue that under these conditions, the **Taylor polynomial**

$$P_{n-1}(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} \tag{64}$$

is a good approximation of f when n is large and $x \in [a, b]$. More formally, prove that for all $\epsilon > 0$, and for all $x \in [a, b]$ there exists δ and N such that if y is a point in $[a, b]$ that is within δ of x (i.e., $|x - y| < \delta$) then $|P_{n-1}(y) - f(x)| < \epsilon$ for all $n \geq N$.

Exercise 53. Recall from the Weierstrass Theorem that because it is continuous, f attains a maximum and a minimum on the interval $[a, b]$. Use the mean value theorems to show that there is a number $c \in [a, b]$ such that

$$R_n = f^{(n)}(c)\frac{(b-a)^n}{n!}. \tag{65}$$

What does this say for (61)?

Theorem 28. **(Taylor's theorem in $\mathbb{R}^n$)** *Let $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is an open set. If f is C^1 then for any $x \in S$ and $y \in S$, we can write*

$$f(y) = f(x) + \nabla f(x)(y - x) + R_1(x, y), \tag{66}$$

where R_1 is a function with the property

$$\lim_{y \to x} \left(\frac{R_1(x, y)}{||x - y||} \right) = 0. \tag{67}$$

If f is C^2 then for any $a \in S$ and $b \in S$, we can write

$$f(y) = f(x) + \nabla f(x)(y - x) + \frac{1}{2}(y - x)' Hf(x)(y - x) + R_2(x, y), \tag{68}$$

where $Hf(x)$ is the Hessian matrix and R_2 is a function with the property

$$\lim_{y \to x} \left(\frac{R_2(x, y)}{||x - y||^2} \right) = 0, \tag{69}$$

and where $\lim_{y \to x}$ is the abusive limit for the convergence of vectors (i.e., the limit holds for all sequences of vectors $\{x_n\}$ that converge to vector y such that $x_n \neq y$ for all n).

Exercise 54. Write down three terms of the Taylor series and evaluate the it at $(x, y) = (1, 1)$ for $f(x, y) = \log(xy)$.