

# Superfluid Condensed Matter Journal Club Talk: Vortices

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## I. RECAPITULATION OF PREVIOUS RESULTS

Recall what we have already learned: That a moving superfluid (which we will often refer to as “helium”) may be described to some extent by the wavefunction[1]

$$|\Psi\rangle = \exp\left(i \sum_{j=1}^N s(\mathbf{r}_j)\right) |0\rangle \quad (1)$$

where the  $\mathbf{r}_i$  are coordinate operators of the helium atoms and the  $|0\rangle$  is the quiescent ground-state of the liquid. There are other approximate wavefunctions which one could use, but we will be satisfied with this one. Given the wavefunction of Eq. (1), we may calculate an associated current-density and mass-density. The current-density and mass-density are found to be proportional which then leads to the relationship

$$\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m} \nabla s(\mathbf{r}) \quad (2)$$

where  $\mathbf{v}_s$  is the velocity of the superfluid. Similar quantum mechanical considerations show that the energy is given by just what we would expect from macroscopic hydrodynamics:

$$E = \frac{1}{2} \rho \int d^3x |\mathbf{v}_s|^2. \quad (3)$$

It should, of course, be noted that there are potential energy corrections to the above (and in general other small corrections), but we are not interested in these corrections for reasons which will be explained later.

From Eq. (2) we see that we have potential flow

$$\nabla \times \mathbf{v}_s = 0. \quad (4)$$

Furthermore, the singlevalued-ness of the superfluid wavefunction leads to the condition that

$$\oint_C \mathbf{v}_s \cdot d\mathbf{l} = 2\pi \frac{\hbar}{m} n \quad (5)$$

where  $n$  is an integer and  $C$  is any closed curve. The above, when combined with Eq. (4), shows that  $n = 0$  in any simply-connected region; since  $\nabla \times \mathbf{v} = 0$  throughout the whole of the simply-connected region this means that the line integral can be turned into a surface integral by Stoke’s Theorem which is then manifestly zero. The integer  $n$  may be different from zero in a multiply-connected region, but still, any paths which are topologically equivalent (can be continuously transformed from one to the other) must have the same value of  $n$ . See Fig. (1) for pictures of multiply- and simply-connected regions and also paths through those regions.

A multiply-connected region may be created by man, macroscopically—for example, if one puts one’s superfluid in a torus shaped bucket. Or, a multiply-connected region may be created spontaneously out of a physical necessity—for example, a vortex line appears. Vortices and vortex lines (and rings) are the main topic of this talk.

A straight vortex line (drawn as a point in 2-d, see Fig. (2)) is a line where the fluid velocity is singular (thus  $\partial_i \partial_j \neq \partial_j \partial_i$  there and  $\nabla \times \mathbf{v}_s$  is not necessarily zero) and where, throughout the rest of the fluid the velocity distribution is given by

$$\mathbf{v}_s = \frac{\hbar}{m} \frac{1}{r} \hat{\theta} = \nabla \left( \frac{\hbar}{m} \theta \right).$$

[... Fig. (4) shows how the gradient of a phase give the velocity distribution for a more complicated distribution than just the gradient of theta, but which has same winding number[2] (see next sections for def. of winding number)].

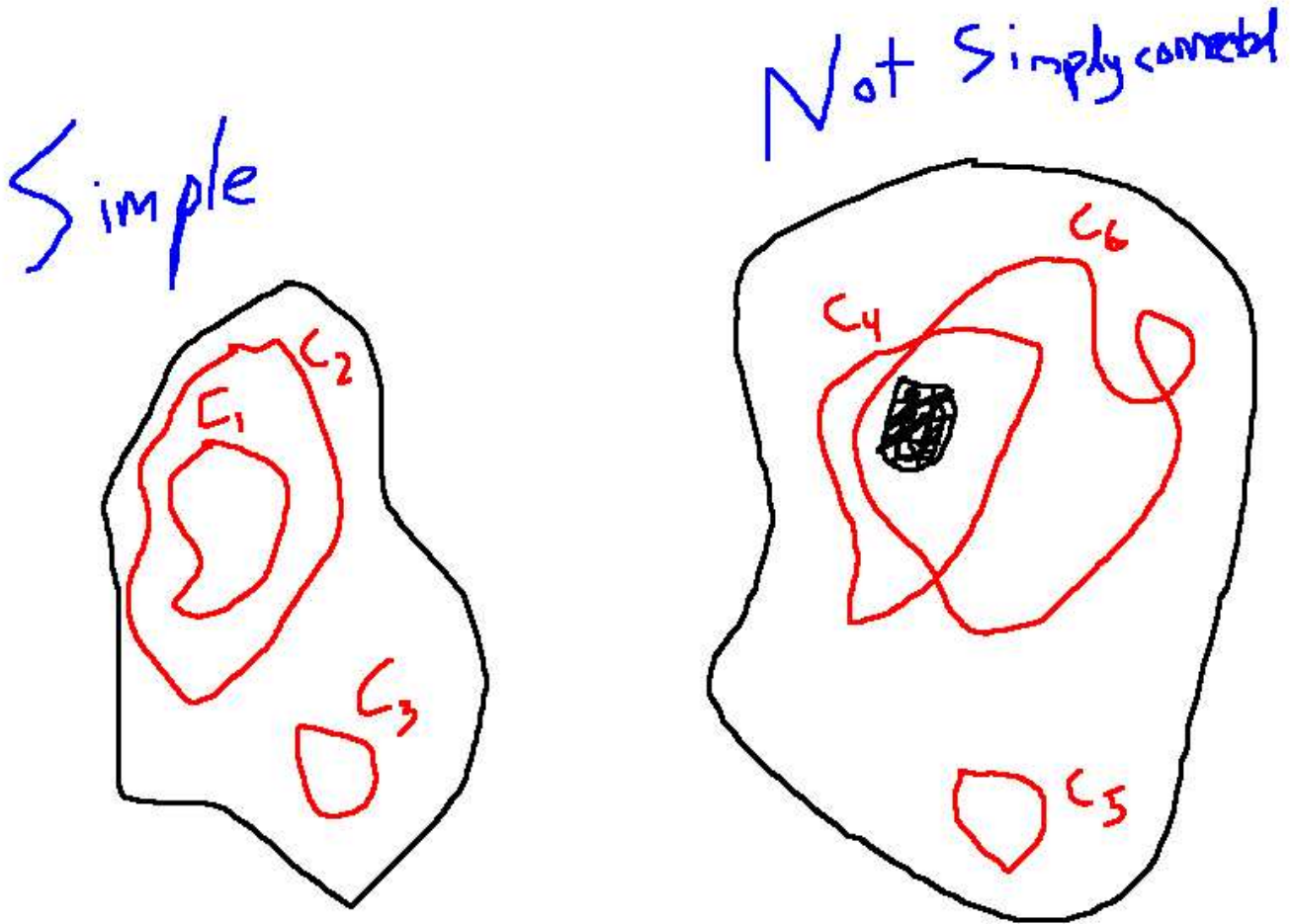


FIG. 1: Simply and multiply connected regions.

This is a canonical and simple starting example of vortex fluid flow. The above velocity distribution happens to have zero curl everywhere in space except at the vortex line thus it is called potential flow and we will accept it as a valid superfluid velocity distribution (even though this distribution blows up at the core). The “core” is the region along the actual vortex line where the velocity apparently blows up. We will take the core to have some small finite radius (microscopic scale) which we will take as a few angstroms.

Throughout this talk we will treat our superfluid as incompressible. Thus the continuity equation tells us

$$\nabla \cdot \mathbf{v}_s = 0 .$$

This will simplify our analysis somewhat as now we are basically interested in solving the Laplace equation in two dimensions since

$$\nabla^2 s(\mathbf{r}) = 0 .$$

Furthermore, we may use our intuition developed from magnetostatics (to some degree) by thinking of  $\mathbf{v}_s$  as analogous to the magnetic field  $\mathbf{B}$  and vortex lines as analogous to current carrying wires. This analogy will be developed further and more precisely in a moment. But first we would like to introduce some experimental observations which we may at first find paradoxical, but we will eventually be able to explain the observations by invoking vortices. The experiment we now discuss was first performed by Osborne[3] and his famous “rotating bucket.” Little known fact: After his career in physics fizzled, Osborne and his “famous rotating bucket” went on to great success as the opening act for an off-off-Broadway production which featured, as a main act, The Amazing “Tapping Pigeons” (a

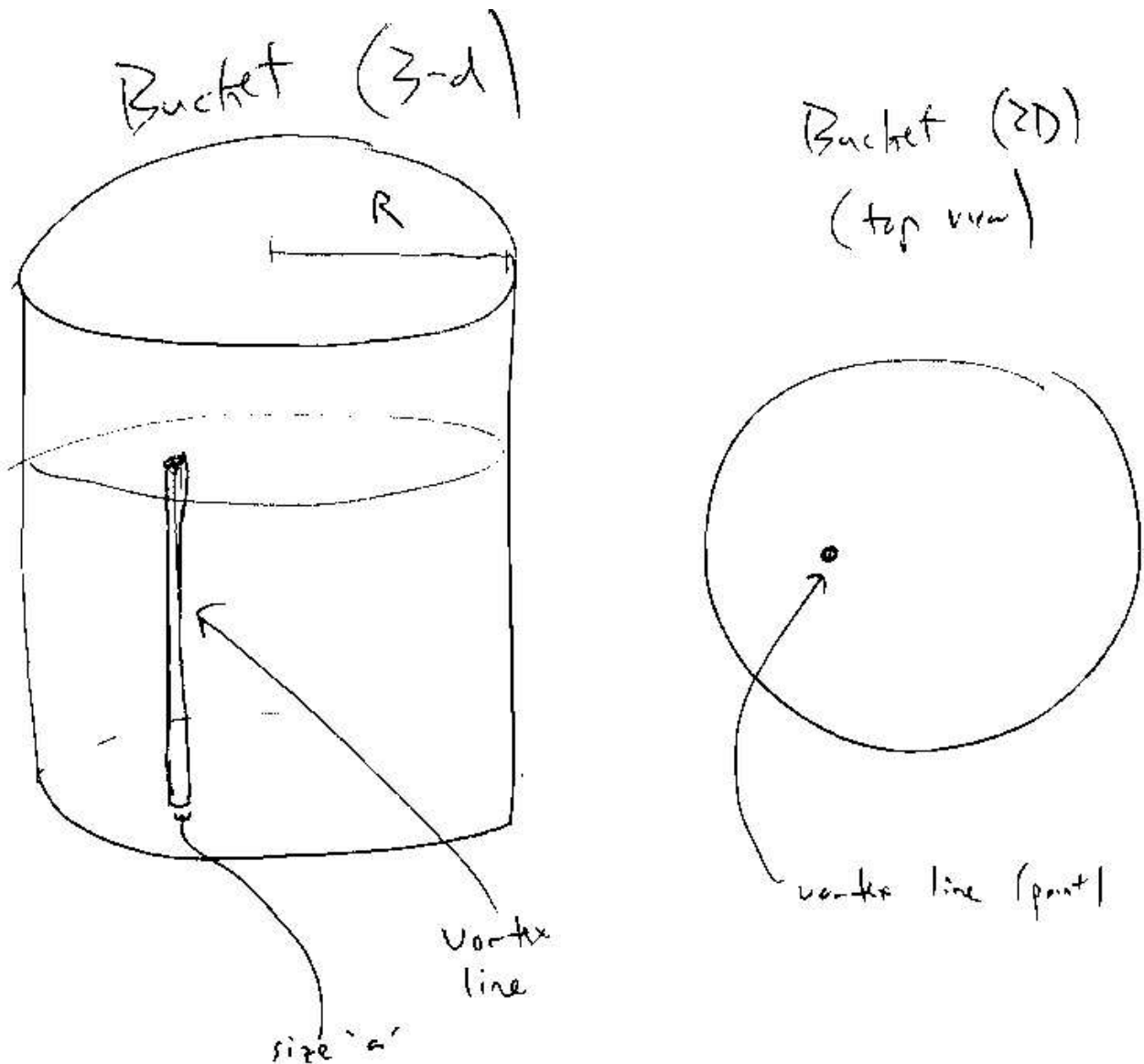


FIG. 2: A bucket with a vortex line in it. Two different views.

group of three trained pigeons who would tap out The Star Spangled Banner on a baby grand piano with their beaks). Many years later, all three pigeons committed suicide by drowning themselves in Osborne's "famous bucket."

## II. OSBORNE'S ROTATING BUCKET

Because the superfluid is perfect it should not begin to rotate even though it may be placed in a bucket, and the bucket forced to rotate. A perfect fluid doesn't care about the tangential component of the velocity at the boundaries with the bucket. The normal fluid in the two-fluid model, however, is not perfect. The normal fluid is viscous and thus it will be dragged along by the bucket and eventually it will act tangentially upon itself until, throughout the bucket, the whole of the normal fluid is rotating just like a rigid body would with velocity  $\omega \times \mathbf{r}$  where  $\omega$  is the angular velocity of the rotating bucket.

One may determine the shape of the surface of the fluid in the rotating bucket within the two-fluid model (see Fig. (3)). The "correct" result can be simply obtained via Freshman Physics if we consider only the normal fluid to be accelerated at a rate  $a = v^2/r = \omega^2 r$ . The whole mass of the fluid (super and normal) is acted upon by gravity. Then

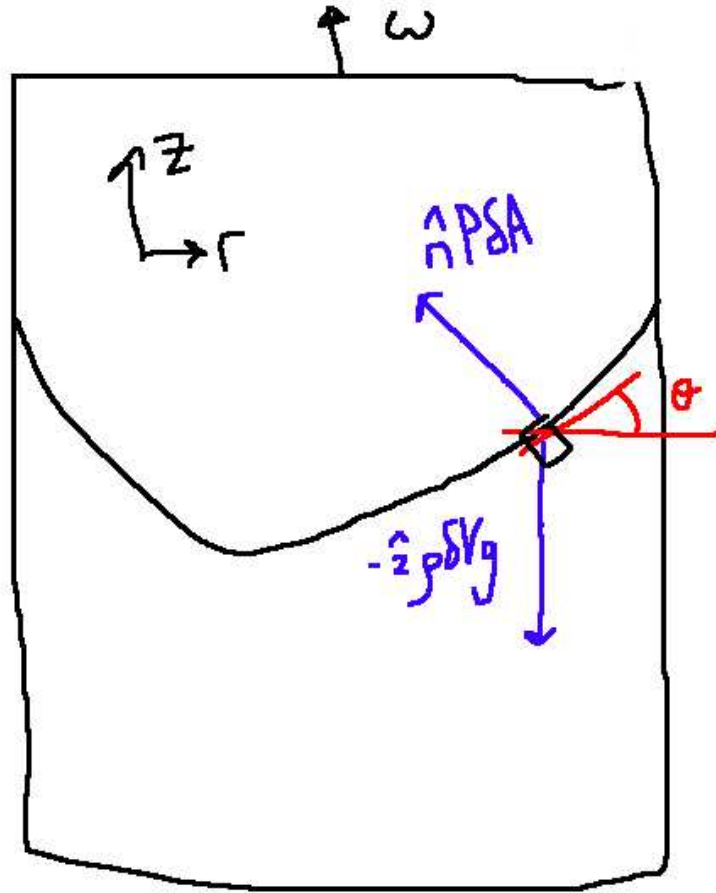


FIG. 3: A rotating bucket.

Newton's laws are applied to a small volume  $\delta V$  of surface area  $\delta A$  on the surface of the fluid which results in two equations: One equation setting the z-components of the force to zero

$$P \delta A \cos \theta = \rho \delta V g$$

where  $P$  is the pressure,  $\rho$  is the total density,  $\theta$  is the angle that the surface makes with the "horizontal" (N.B.,  $\tan \theta = dz/dr$ ), and  $g$  is the acceleration due to gravity; and one equation setting the r-components of the force to equal  $\rho_n \delta V \omega^2 r$

$$P \delta A \sin \theta = \rho_n \delta V \omega^2 r .$$

Combining the above two equations gives

$$\frac{dz}{dr} = \tan \theta = \frac{\rho_n \omega^2 r}{\rho g} .$$

Or rather,

$$z(r) = \frac{\rho_n \omega^2 r^2}{\rho 2g} .$$

The shape of the surface is a parabola, just like in the case of a regular fluid, except that the curvature is diminished by the factor of  $\rho_n/\rho$  which goes to zero as the temperature goes to zero. The curvature of the surface of a spinning

superfluid should be less than that of a regular fluid with the same density. . . Unfortunately, Osborne did not observe this “correct” result. Osborne observed that the shape of the fluid surface in the rotating bucket was given by

$$z(r) = \frac{\omega^2 r^2}{2g}$$

just as we would expect for a regular fluid. Hmmm. This, sadly, lead to a paradox. Specifically, the superfluid appears to have a non-zero vorticity of  $2\omega\hat{z}$ ; a fluid which rotates like a rigid body has a velocity given by  $\mathbf{v} = \omega \times \mathbf{r}$  thus

$$(\nabla \times \mathbf{v})_i = \epsilon_{ijk} \nabla_j \epsilon_{klm} \omega_l r_m = \epsilon_{ijk} \epsilon_{lmk} \delta_{jm} \omega_l = 2\omega_i .$$

And so the fluid is not irrotational, contradicting the fact that the velocity is the gradient of a scalar. So, what gives?

### III. VORTEX LINES

The way out of the paradox of Section II is via the introduction of vortex lines. We suppose that only at isolated points (lines in 3-d) the condition of irrotationality is violated, thus throughout the bulk of the superfluid we maintain zero vorticity. Such a velocity distribution is that of a vortex line. The region where the vorticity blows up is thought of as an infinitesimally thin line (or point in 2-d) known as the “core.” We will actually consider the core to have a microscopic radius of a few angstroms. This is sensible within the atomic picture. Thus  $\nabla \times \mathbf{v}$  is delta-function like, as we will see in more detail shortly.

Thus we are motivated to study velocity flows such as

$$\mathbf{v}_s = \frac{\hbar}{m} \frac{1}{r} \hat{\theta} = \nabla \left( \frac{\hbar}{m} \theta \right) .$$

. . . which I already said. . . Whatever.

More generally, I might consider a velocity distribution

$$\mathbf{v} = \nabla \alpha(\theta)$$

where alpha is some function of theta. (Even more generally, I can, of course, consider alpha to be a function of  $r$  and  $\theta$ , but for simplicity we only consider a function of theta.) We see that  $\alpha = \theta$  gives us unit positive “charge” vortices and that  $\alpha = -\theta$  gives us unit negative “charge” vortices. The “charge” of the vortex is often referred to as “winding number” because that is what it is. If I think of any function of theta which turns thru  $2\pi$  as theta does then I map a circle into a circle winding around once. All maps which do this are homotopically equivalent and they have the same winding number which is a topological invariant. See Fig. (4) for a picture. N.B., the particular function  $\alpha(\theta)$  shown in the picture does not satisfy the incompressibility condition that  $\nabla \cdot \mathbf{v} = \mathbf{0}$ ; one can show that the only functions  $\alpha(\theta)$  of theta alone that satisfy the divergenceless condition are linear; this fact is not of any importance to us, I just wanted to point it out for the sake of trivia.

For reasons which may be easily understood once we have calculated the energy of various vortex configurations we will, for the most part, be interested only in vortices of charge plus or minus one. These alone are quite interesting enough.

Vortex flow is special because it is curl-less everywhere except at the core. For the simple example above we can explicitly calculate the line integral about a circular path if we would like to bore ourselves. Of course the integral around any path that goes once around the core must be equal to the integral around this circular path since  $\nabla \times \mathbf{v}$  integrated along the area described by the difference in paths is manifestly zero. For a circular path of radius  $R$  we have

$$\oint \mathbf{v} \cdot d\mathbf{l} = \int_0^{2\pi} d\theta R \hat{\theta} \cdot \hat{\theta} \frac{\hbar}{mR} = 2\pi \frac{\hbar}{m} .$$

By converting this to a surface integral we see explicitly that the curl must have a delta-function-like behavior since

$$\frac{\hbar}{m} 2\pi = \int_A dx dy \hat{z} \cdot (\nabla \times \mathbf{v}) .$$

And since the area  $A$  may be shrunk down as small as we like we must have

$$\nabla \times \mathbf{v} = \hat{z} \delta(x) \delta(y) \frac{2\pi\hbar}{m} .$$

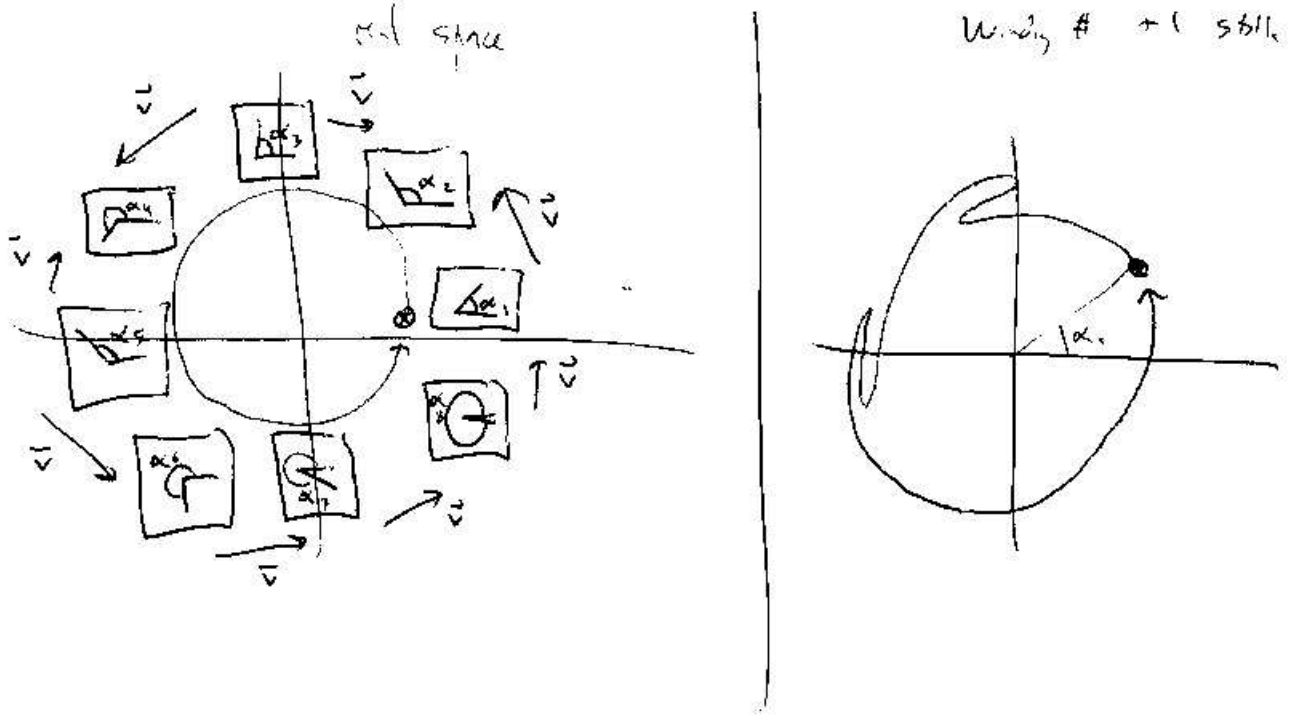


FIG. 4: A mapping of winding number +1.

More generally, if I have any curvy line vortex I can zoom in on one small part of it and I will see a straight line vortex and I may repeat exactly the procedure I just applied (only this time I will call my  $z$ -axis the  $\eta$ -direction where  $\hat{\eta}$  points along the vortex line) to find

$$\nabla \times \mathbf{v}_s = \frac{2\pi\hbar}{m} \hat{\eta} \delta^2(\mathbf{x}_\perp).$$

Because of the divergenceless condition I can now work in exact analogy with magnetostatics to write down an integral equation for the velocity distribution at any point in space as follows:

$$\nabla \times \nabla \times \mathbf{v} = \nabla \times \left( \frac{2\pi\hbar}{m} \hat{\eta} \delta^2(\mathbf{x}_\perp) \right) = -\nabla^2 \mathbf{v}.$$

Thus

$$\begin{aligned} \mathbf{v}(\mathbf{r}) &= \int d^3 r' \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \nabla \times \frac{2\pi\hbar}{m} \hat{\eta} \delta^2(\mathbf{r}'_\perp) \\ &= \frac{2\pi\hbar}{4\pi m} \int d^3 r' \hat{\eta} \delta^2(\mathbf{r}'_\perp) \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \\ &\equiv \frac{\hbar}{2m} \int d\mathbf{l} \times \frac{\mathbf{R}}{R^3}. \end{aligned} \quad (6)$$

The 2nd to last step is accomplished via integration by parts and the last step defines the vector  $\mathbf{R}$  from the point on the path to the point where  $\mathbf{v}$  is being calculated. Explicitly, in terms of a parametrized path  $\mathbf{l}(s)$  where  $s$  is a real parameter, we have:

$$\mathbf{v}(\mathbf{r}) = \int ds \frac{d\mathbf{l}}{ds} \times \frac{(\mathbf{r}-\mathbf{l}(s))}{|\mathbf{r}-\mathbf{l}(s)|^3}.$$

Near the (possible quite curvy) vortex line we can “zoom in” and see that the velocity will be given by a part which blows up as  $1/r$  and a part which is smooth and which is given by the velocity as calculated by all other vortex lines and other parts of the line itself. The smooth part of the velocity at the point of the core is the velocity at which that point on the vortex line moves.

For example, a straight vortex line (point) just sitting there by itself shouldn't move at all. If you think it should, then which way?—all directions are equivalent. Placing a vortex in a steady flow, then, the vortex should move with the flow. This, I think, is only approx true, since there will be a force (Magnus) on it which will eventually induce transverse motion... but this force doesn't give us the acceleration, the acceleration is determined by some effective mass of the vortex? which is what? I don't know. Also, there are similar energy considerations to take into account when looking at the motion of vortex pairs. This rule that the vortex moves with the velocity of the flow at the point of the vortex (neglecting the action of the vortex on itself) cannot be completely true, can it? Maybe only so far as the pair cannot lose energy by any mechanism, maybe this holds for  $T=0$ .

### energy of a vortex line

We first consider the energy of an isolated straight vortex line (point) in a spherical bucket of radius  $R$  and height  $L$ . We have seen that the energy per unit length ( $E/L$ ) is given by

$$\begin{aligned} E/L &= \frac{\rho}{2} \int d^2x v_s^2 \\ &= \frac{\rho}{2} \int d^2x \left( \frac{\hbar}{mr} \right)^2 \\ &= \frac{\rho}{2} \left( \frac{\hbar}{m} \right)^2 2\pi \int \frac{dr}{r} \\ &= \rho\pi \left( \frac{\hbar}{m} \right)^2 \log\left(\frac{R}{a}\right) \end{aligned} \quad (7)$$

where  $a$  is the core size. Because we don't know  $a$ , but we do know that  $R$  is much greater than  $a$  to the point that  $\log R$  is much greater than  $\log a$  we say that the energy is determined with “log accuracy.” We are satisfied with this. Of course, there is some potential energy contribution to the total, but it has the form  $E \sim \int d^2x \delta\rho^2 \sim a^2$  or something and we neglect this contribution—rather, it can be shown that the potential contribution is negligible.

We find it useful to define the energy of a single straight vortex line of unit strength as

$$E_1 \equiv \rho\pi \left( \frac{\hbar}{m} \right)^2 \log\left(\frac{R}{a}\right)$$

### motion of pairs of vortices

There were some questions raised during the practice talk about the actual motion of the vortices, and it seems that perhaps it is not exactly what is explained below. So to remedy this let us assume that the vortex pair I am about to discuss are fixed to be a distance  $d$  apart by a rigid, yet massless and totally fluid-permiable, rod. (Or, otherwise assume that there is no mechanism by which the pair can lose energy). Then the motions are as explained in Nozieres and Pines[4] and as explained below.

Each vortex of the pair is considered to be perfectly straight and is sitting a distance  $d$  away from the other. Each vortex moves according to the velocity distribution dictated by the other. For example, two vortices with the same charge will circle around each other (see Fig. (5)) whereas two vortices of opposite charge will move along in a straight path together at the same velocity (see Fig (5)).

We can also treat cute problems such as that of a vortex line close to a wall by introducing an “image vortex” so that the normal component of the velocity is zero and thus we can figure out how the line moves near the wall (see Fig. (6))... even cuter still we can see that a smoke ring[5] should expand (and slow down) as it approaches a wall. But we will see that later.

This same question that we discuss above about what is the true motion of the vortex line can be phrased in terms of vortex rings as well. For example, one can show [6] that a vortex ring moves with a velocity  $v \sim \log(R_0)/R_0$  along

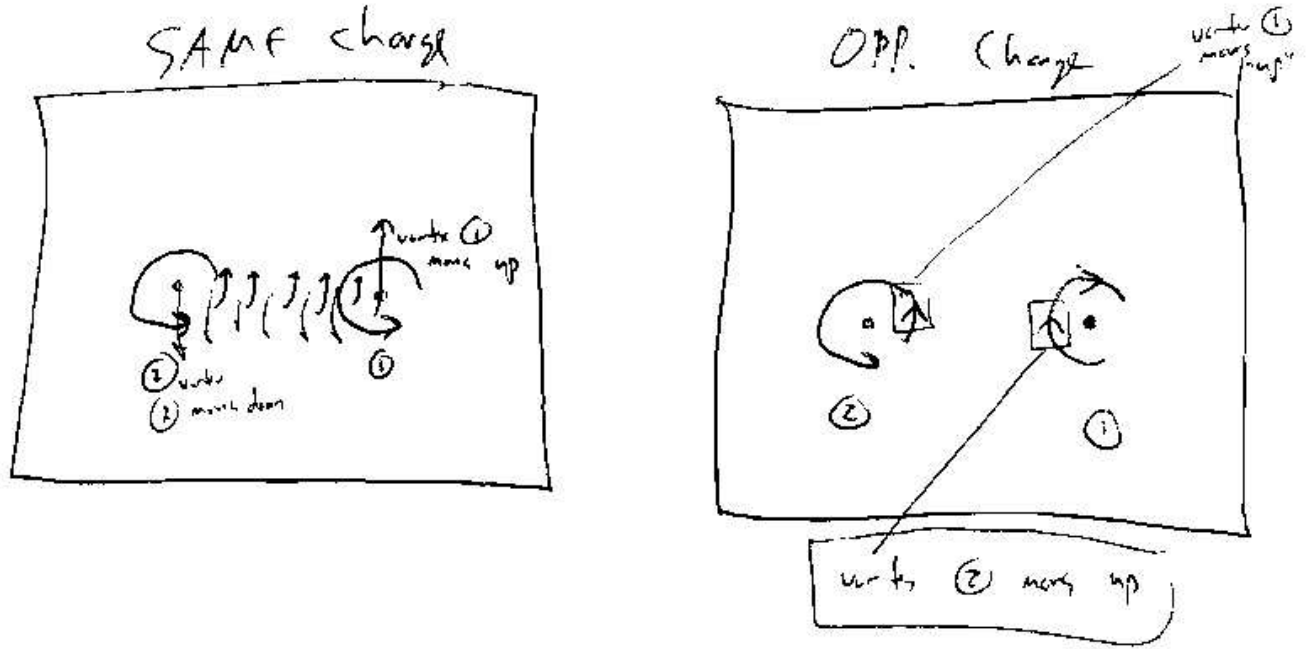


FIG. 5: some vortex pairs and their motions.

its own axis due to the action of other parts of the ring on a point of the ring. This is shown in the same way as above, neglecting the divergent part of the velocity due to one point on the ring. But this also neglects the fact that the ring feels a force which tends to shrink it, does it not? The energy of the ring  $E \sim R_0 \log(R_0)$  decreases as the ring shrinks so it should shrink with some rate and thus increase its velocity at some rate too. eh? (assuming it can lose energy).

#### energy of vortex pairs

We will now calculate the energy of two vortex pairs of opposite charge and of unit magnitude. This calculation is easily generalized to the calculation of the energy of an arbitrary array of charged vortices. The energy of a pair of vortices (one located at  $\mathbf{x}_1$  and the other at  $\mathbf{x}_2$  with  $|\mathbf{x}_1 - \mathbf{x}_2| = d$ ) is given by

$$E/L = \frac{\rho}{2} \int d^2x (\mathbf{v}_1 + \mathbf{v}_2)^2$$

where  $v_1 = \hbar/m\nabla\theta_1$  and  $v_2 = \hbar/m\nabla\theta_2$  with

$$\theta_1 \equiv \tan^{-1} \frac{y - y_1}{x - x_1}.$$

We take  $\theta_2$  to run in the opposite sense than usual and this makes the vortex at  $\mathbf{x}_2$  negatively charged. See Fig. (8) to make sense of all this. Also note that  $\nabla^2\theta_1 = \nabla^2\theta_2 = 0$ ; this can be seen simply by noting that  $\nabla^2\theta = \nabla \cdot \hat{\theta}/r = 0$  and that  $\nabla^2$  doesn't care about being translated in space.

Using the definition of  $E_1$  we have

$$E/L = 2E_1 + \rho \int_M d^2x \mathbf{v}_1 \cdot \mathbf{v}_2$$

where  $M$  is the two dimensional manifold which we are integrating over. I.e., some large circle of radius  $R \rightarrow \infty$  with two small holes of radius  $a \rightarrow 0$  cut out around the vortices. See Fig. (7)

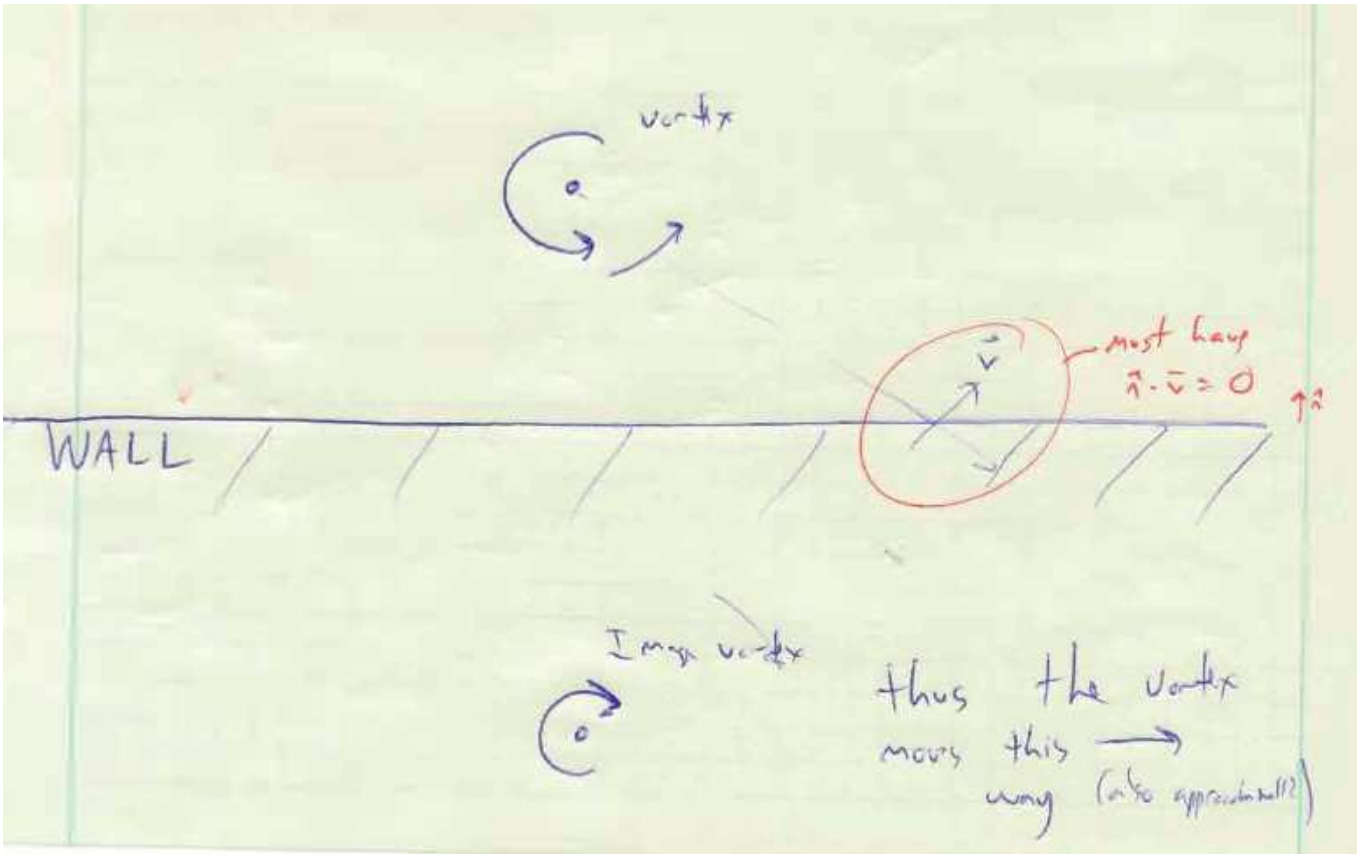


FIG. 6: A vortex line near a wall.

Now, to simply evaluate the interaction energy we note that both of the velocities are continuous throughout  $M$  and so the value of the integral is completely unchanged if we integrate over a new manifold  $M'$  with two cut lines. See Fig. (8) Now we can calculate the interaction energy per unit length ( $E/L - 2E_1$ ) as

$$\begin{aligned}
 E/L - 2E_1 &= \rho \int d^2x \nabla \left( \frac{\theta_1 \hbar}{m} \right) \cdot \mathbf{v}_2 \\
 &= \rho \int d^2x \nabla \cdot \left( \frac{\theta_1 \hbar}{m} \mathbf{v}_2 \right) + 0 \\
 &= \rho \int_{\partial M'} \mathbf{d}\Sigma \cdot \mathbf{v}_2 \frac{\hbar \theta_1}{m} \\
 &= \rho \frac{\hbar}{m} \int_{L_a, L_b} \mathbf{d}\Sigma \cdot \mathbf{v}_2 \theta_1 \\
 &= 0 + 2\pi\rho \left( \frac{\hbar}{m} \right)^2 \int_d^R dr \frac{1}{r} \hat{y} \cdot -\hat{y} \\
 &= -2\pi\rho \frac{\hbar}{m} \frac{\hbar}{m} \log\left(\frac{R}{d}\right). \tag{8}
 \end{aligned}$$

Then, when we combine this with  $2E_1$  we find

$$E/L = 2\pi\rho \left( \frac{\hbar}{m} \right)^2 \log\left(\frac{d}{a}\right)$$

From this equation we can more easily understand our previous complaint about vortex motion. These oppositely charged vortices should attract each other with a force given by the gradient of the pair potential. Thus instead of moving along side by side they should also draw closer to each other (eventually annihilating at some time when  $d$  becomes close to  $a$ ).

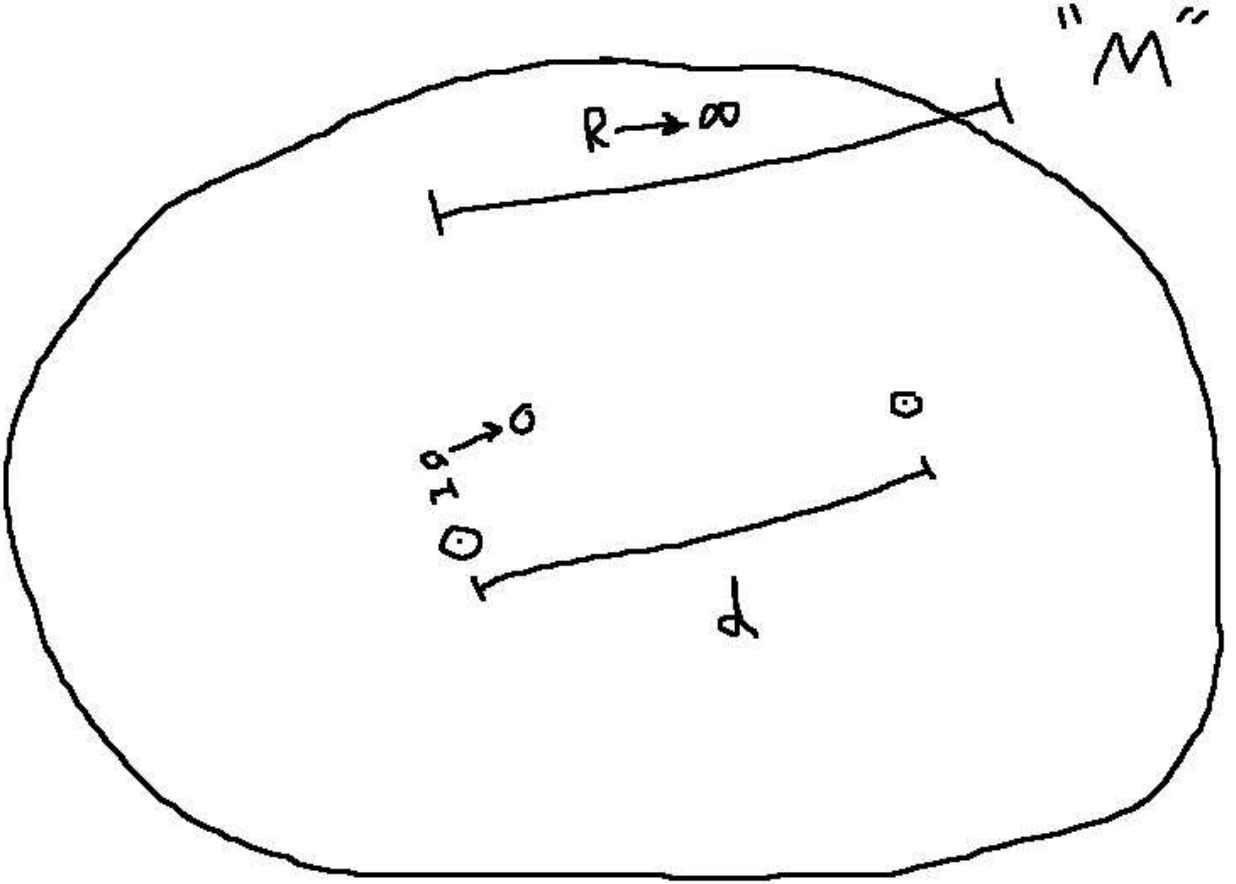


FIG. 7: Region of integration M.

Also note, now that we know the pair potential we have everything we need to evaluate the energy of arbitrary arrays of vortices; they only interact in pairs. If we work through this calculation for vortices of arbitrary charge we find a result which lets us explain why we only care about vortices of unit charge. We find:

$$E/L = \pi\rho \left(\frac{\hbar}{m}\right)^2 \left[ (q_1 + q_2)^2 \log\left(\frac{R}{a}\right) - 2q_1q_2 \log\left(\frac{d}{a}\right) \right]$$

Thus the energy of two vortices each with single units of charge is less than the energy of a single vortex with two units of charge.

Similar calculations can easily give the energy of more complicated vortex arrays—For example, four vortices arranged in a diamond shape. The result is just evaluated in exactly the same way, but we have a sum over interaction energies of each pair and each pair may be considered completely separately. And we also include the energy of each pair alone. For an array of  $N$  vortices where the charge of the  $i$ th vortex is  $q_i$  we have

$$E/L = \pi\rho \left(\frac{\hbar}{m}\right)^2 \sum_{i,j=1}^N q_i q_j \log\left(\frac{R}{d_{ij}}\right)$$

where  $d_{ij}$  is the distance between the  $i$ th and  $j$ th vortex for  $i \neq j$  and  $d_{ii} \equiv a$ .

For example, for the diamond shape with total charge zero shown in Fig. (9), we find an energy

$$2\pi\rho \left(\frac{\hbar}{m}\right)^2 \left( n^2 \log\left(\frac{d_{12}}{a}\right) + m^2 \log\left(\frac{d_{34}}{a}\right) \right).$$

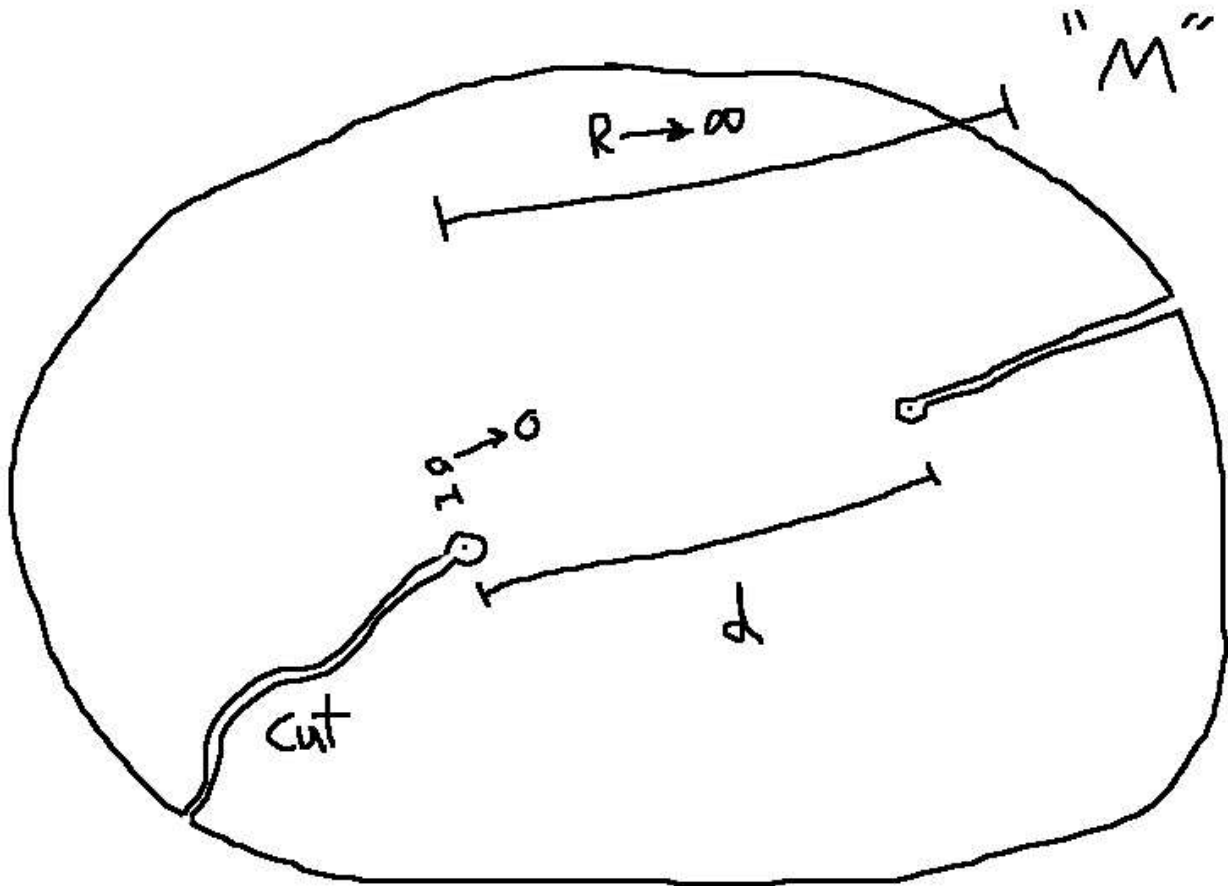


FIG. 8: Region of integration  $M'$  (oops, it still says "M" in the picture).

#### IV. FEYNMAN'S CRITICAL VELOCITY CALCULATION

Following Feynman we can use what we have learned about the energy of vortex pairs in order to estimate the true critical velocity of a superfluid. Recall that, long ago, we estimated the critical velocity by considering the energy needed to create rotons. This estimate was two orders of magnitude too high. We will now estimate the critical velocity by considering how much energy is needed to create vortex pairs when a superfluid jets out into a quiescent vat of fluid.

The superfluid can in theory maintain potential flow as it flows out of a pipe of size  $d$  into a large vat. A normal fluid, however, would just jet out and create regions of nonzero vorticity. See Fig. (10). And we can mimic this in a superfluid by using vortices in the same way that a solinoid makes a magnetic field through the center but not outside for tight coil spacing. By using vortices the superfluid flow remains "potential flow" except for isolated singularities.

If the jet shoots out with velocity  $v_0$  into the fluid then the circulation about a narrow long loop of size  $L$  is  $v_0 L$ . The integral about a loop containing vortices spaced out by a distance  $x$  gives a circulation  $2\pi\hbar L/(xm)$ . Thus we must have

$$\frac{1}{x} = \frac{v_0 m}{2\pi\hbar}.$$

These vortex pairs flow out of the orifice at a velocity of  $v_0/2$  and so a vortex pair is created every  $T$  seconds where

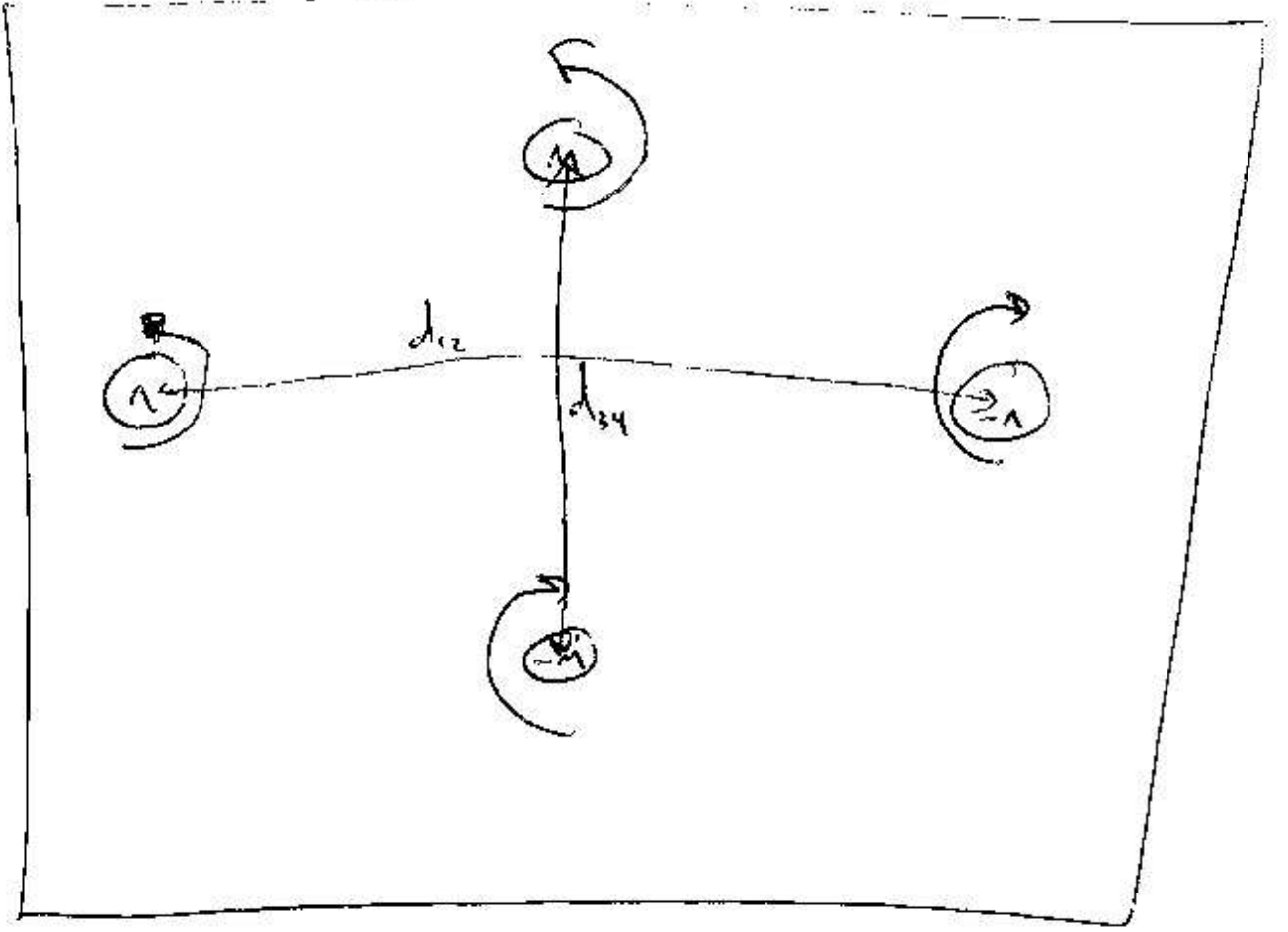


FIG. 9: A diamond-shaped array of four vortices having charges  $+n, -n, +m,$  and  $-m$ .

$T = x/(v_0/2)$ . The energy of a single pair is given by

$$E_{pair} = 2\pi\rho\left(\frac{\hbar}{m}\right)^2 \log\left(\frac{d}{a}\right)$$

thus the energy flowing out of the orifice (per unit length  $L$ ) per time  $T$  used to create vortices is given by

$$E'/(LT) = \frac{E_{pair}}{L} \frac{v_0}{2x} = \frac{v_0}{2} \frac{v_0 m}{2\pi\hbar} 2\pi\rho\left(\frac{\hbar}{m}\right)^2 \log\left(\frac{d}{a}\right).$$

But the available energy in the fluid (which flows out a distance  $v_0 T$  in time  $T$ ) is

$$E/(v_0 T L d) = \frac{1}{2}\rho v_0^2.$$

Or rather

$$E/(LT) = v_0 d \frac{1}{2}\rho v_0^2.$$

The above can supply the energy to create vortices at a critical velocity (obtained by equating  $E/LT$  to  $E'/LT$ )

$$v_c = \frac{\hbar}{dm} \log\left(\frac{d}{a}\right).$$

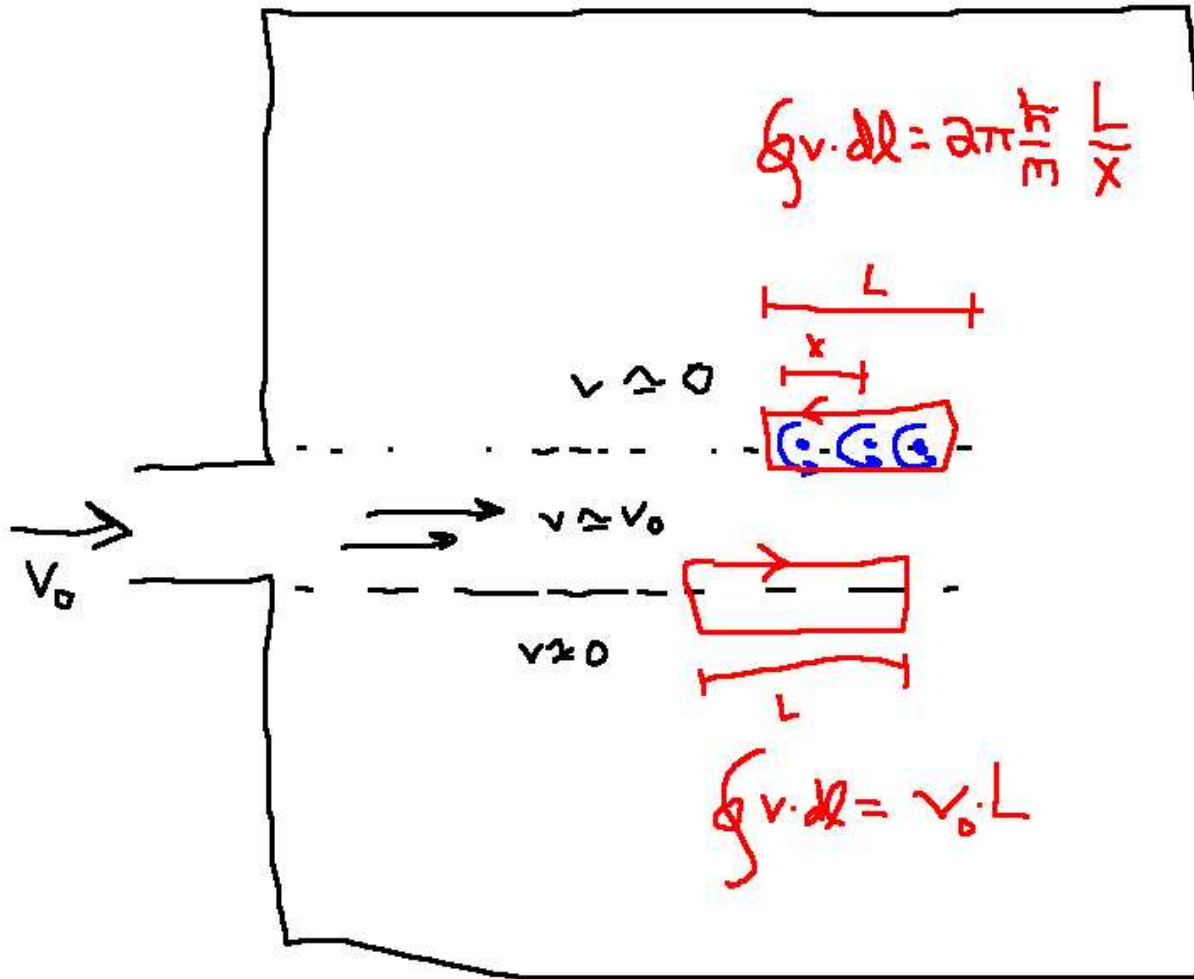


FIG. 10: A vat with a jet of fluid shooting out into it.

Interesting side note: This is about twice the velocity of a vortex ring (which I have not yet defined... sorry, I think I'm skipping that part... for more info see this[7] reference.) of radius  $d$ . Can also get this result by calculating energy and momentum of a vortex ring and using  $E - pv_c = 0$  as our original estimate of the critical velocity.

## V. BUCKET PARADOX RESOLUTION

This is easy. We have seen that it is energetically favorable to have a bunch of unit strength vortices rather than one big vortex, so we will not be surprised to find this situation in a rotating bucket. We might wonder about whether there would be vortices created at all in a rotating bucket and we can estimate a critical angular velocity above which the superfluid will find it favorable to violate the condition of zero-curl at isolated point. This critical velocity is small and for a bucket rotating at a reasonable rate we expect lots of vortices, lots of unit vortices that want to be approx evenly spaced. That is, we should see, in the rotating bucket an approximately uniform surface density  $n_0$  of vortex points. Thus the line integral around a circle in the bucket of radius  $r$  is given in terms of the number of vortices enclosed as

$$\oint_{C_r} \mathbf{dl} \cdot \mathbf{v} = \frac{2\pi\hbar}{m} N_v = \frac{2\pi}{\hbar} n_0 \pi r^2.$$

But this line integral is also given by  $v(r)2\pi r$  which gives us an observed velocity distribution of

$$v(r) = \frac{\hbar n_0 \pi}{m} r .$$

which is exactly what we get for rigid body rotation at a constant angular velocity  $\omega$  (linear increase in velocity with  $r$ ) for  $\omega = \frac{\hbar n_0 \pi}{m}$ .

Thus by adding up a bunch of velocity distributions which all individually fall off as  $1/r$  we create a velocity distribution which macroscopically appears to increase as  $r$  since the falloff is compensated for by the large number of vortices within a given circle.

Because the velocity increase linearly the superfluid should have a parabolic surface just like Osborne observed.  
The End.

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