CS 366: Computational Social Choice	**Winter 2024**

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4.1 Participatory Budgeting

One application of computational social choice is *participatory budgeting*. In participatory budgeting, members of a community contribute to the construction of a budget by directly voting (or otherwise expressing their preferences) on how public money should be spent. This stands in contrast with traditional budgeting processes where bureaucrats or elected representatives decide the use of public funds without direct input from constituents. Participatory budgeting schemes have quickly increased in popularity since their introduction in Brazil in 1988 [Wampler, 2000]. Today, participatory budgeting schemes are used in over 7,000 municipalities across the world.[1] [2]

But how exactly do citizens express their budgeting preferences, and how should these preferences be aggregated to produce a budget? In this lecture, we build a framework that will allow us to study these questions mathematically. We discuss a social choice rule, *knapsack voting*, that is strategy-proof for the participatory budgeting problem (under certain assumptions). We also describe a linear programming formulation of the participatory budgeting problem. We conclude by outlining some potential fairness issues with participatory budgeting.

4.2 A Mathematical Framework for Participatory Budgeting

Suppose that we have a set of N voters, who must decide how a budget of size $B > 0$ is allocated across M expense items. Each expense item j has a fixed cost of $c_j \geq 0$, for $j = 1, \ldots, M$. Our goal is to produce an allocation $\mathbf{x} = (x_1, \ldots, x_M)$, that assigns funds x_j to expense item j. We say that an allocation $\mathbf{x} \in \mathbb{R}^M$ is *feasible* if it satisfies the following criteria:

1. $\mathbf{x} \geq 0$
2. $x_j \leq c_j$ for all $j = 1, \ldots, M$
3. $\sum_{j=1}^{M} x_j \leq B$

In other words, an allocation is feasible if it assigns a non-negative amount of money to each expense item, without exceeding the total budget or over-funding any individual expense item.

However, we still need methods to elicit voter preferences and aggregate them. There are many ways to do this, each with its own benefits and drawbacks. The process of elicitation and aggregation is a choice that depends on what qualities we would like the social choice function to have. A comprehensive discussion of approaches to this problem can be found in Aziz and Shah [2021].

[1] I thank the UC Berkeley EECS Department for their lecture notes template.
[2] See `https://www.participatorybudgeting.org/about-pb/`

One potential elicitation method is as follows. Suppose each voter i has an ideal budget $\mathbf{z^{(i)}} \in \mathbb{R}^M$, such that $0 \leq z_j^{(i)} \leq c_j$ for all $j = 1, \dots, M$ and $\sum_{j=1}^M z_j^{(i)} = B$. We now discuss an aggregation method called *knapsack voting.*

4.3 Knapsack Voting

Suppose that the cost c_j of each expense item is an integer, and that each voter i's ideal budget allocations $z_j^{(i)}$ are all also integers, as is the budget B. We divide the cost c_j of each item into c_j dollars, which we denote $D_1^{(j)}, \dots, D_{c_j}^{(j)}$. Then given an ideal allocation $\mathbf{z^{(i)}}$, we define a voter's approval set as $S_i \equiv \bigcup_{j=1}^M \{D_1^{(j)}, \dots, D_{z_j^{(i)}}^{(j)}\}$. In other words, if voter i wants $z_j^{(i)}$ dollars allocated to expense item j, we say i approves of "candidates" $D_1^{(j)}$ through $D_{z_j^{(i)}}^{(j)}$. We define the full set of dollars as $\mathcal{P}$.

Example
Suppose we have a total budget $B = 10$, $M = 3$ expense items, and a cost vector $\mathbf{c} = (2, 5, 7)$. The ideal allocation of voter i is given by $\mathbf{z^{(i)}} = (2, 3, 5)$. Then voter i's approval set is given by

$$S_i = \{D_1^{(1)}, D_2^{(1)}, D_1^{(2)}, D_2^{(2)}, D_3^{(2)}, D_1^{(3)}, D_2^{(3)}, D_3^{(3)}, D_4^{(3)}, D_5^{(3)}\}$$

Given this setup, we define the knapsack voting social choice function as follows.

Definition 4.1 (Knapsack Voting) *From Goel et al. [2019]*

- *Each voter i submits a subset $S_i \subset \mathcal{P}$ such that $|S_i| = B$.*
- *The winning set is given by $\arg\max_{S:|S|=B} \sum_{y \in S} score(y)$, using a consistent deterministic tiebreaking rule.*

where $score(y) = |i \in \{1, \dots, N\} \ : \ y \in S_i|$.

Knapsack voting will select the B dollars that receive the most approvals. Note the key fact that a voter who votes for $D_{k+1}^{(j)}$ must also have voted for $D_k^{(j)}$.

To show strategy-proofness, we require a utility model, which will tell us how much a given voter benefits from a budget allocation $\mathbf{x}$. For this we introduce the L^1-distance cost function.

Definition 4.2 (L^1-distance Utility) *The utility a voter i with ideal allocation $\mathbf{z^{(i)}}$ obtains from an allocation $\mathbf{x}$ is given by*

$$u_i(x) = -\|\mathbf{z^{(i)}} - \mathbf{x}\|_1 = -\sum_{j=1}^M \left| z_j^{(i)} - x_j \right|$$

The L^1-distance utility function has the appealing interpretation that the (dis-)utility of a voter is given by the total amount of "mis-allocated" money. Goel et al. [2019] show that under the L^1-distance utility model, knapsack voting is strategy proof,

4.4 Other Utility Models

Some other possible voter utility functions we can impose include

- **Linear utility**: $u_i(\mathbf{x}) = \sum_{j=1}^{M} u_j^{(i)} x_j$. Voter i gets utility $u_j^{(i)}$ for each additional dollar spent on budget item j.
- **Overlap utility**: $u_i(\mathbf{x}) = \sum_{j=1}^{M} \min(x_j, z_j^{(i)})$. Voter i has ideal budget $\mathbf{z_i} = (z_1^{(i)}, \dots, z_M^{(i)})$, and gets utility from budget spending on item j up to ideal value $z_j^{(i)}$ but no utility from any additional spending.

The overlap utility is closely connected to the L^1-distance utility we proposed earlier. In fact, we can show that under the additional assumption that $\sum_{j=1}^{M} x_j = B$ (the budget uses all available funds), then the overlap utility function is an affine transformation of the L^1-distance utility function.

Claim 4.3 *If $\sum_{j=1}^{M} x_j = B$, then*

$$\sum_{j=1}^{M} \min(x_j, z_j^{(i)}) = B - \frac{1}{2}\|\mathbf{z^{(i)}} - \mathbf{x}\|_1$$

where B is the total budget.

Proof: Assume without loss of generality that $x_j \geq z_j^{(i)}$ for $j = 1, \dots, k$ and $x_j \leq z_j^{(i)}$ for $j = k+1, \dots, M$. First observe that

$$\|\mathbf{z^{(i)}} - \mathbf{x}\|_1 = \sum_{j=1}^{k}(x_j - z_j^{(i)}) + \sum_{j=k+1}^{M}(z_j^{(i)} - x_j) \iff$$

$$\|\mathbf{z^{(i)}} - \mathbf{x}\|_1 = \left(B - \sum_{j=k+1}^{M} x_j\right) - \left(B - \sum_{j=k+1}^{M} z_j^{(i)}\right) + \sum_{j=k+1}^{M}(z_j^{(i)} - x_j) \iff$$

$$\frac{1}{2}\|\mathbf{z^{(i)}} - \mathbf{x}\|_1 = \sum_{j=k+1}^{M}(z_j^{(i)} - x_j)$$

Using this, we can conclude that

$$\begin{aligned}
\sum_{j=1}^{M} \min(z_j^{(i)}, x_j) &= \sum_{j=1}^{k} z_j^{(i)} + \sum_{j=k+1}^{M} x_j \\
&= B - \sum_{j=k+1}^{M}(z_j^{(i)} - x_j) \\
&= B - \frac{1}{2}\|\mathbf{z^{(i)}} - \mathbf{x}\|_1
\end{aligned}$$

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This result is important because it means that an optimal solution to the participatory budgeting problem for the L^1 distance utility function will be an optimal solution for the problem using the overlap utility function. This will allow us to pose the participatory budgeting problem as a linear programming problem.

4.5 Linear Programming Formulation

If our objective to maximize the sum of all individual voter utilities, then our participatory budgeting/knapsack voting problem is

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} \sum_{j=1}^{M} \min(z_j^{(i)}, x_j) \\ \text{subject to} \quad & \sum_{j=1}^{M} x_j = B \\ & x_j \leq c_j \;\; (\forall j) \\ & x_j \geq 0 \;\; (\forall j) \end{aligned}$$

The program as formulated above is not linear due to the $\min(\cdot)$ functions. To address this, we introduce MN new decision variables denoted $\mu_j^{(i)}$ and impose the constraints $\mu_j^{(i)} \leq x_j$ and $\mu_j^{(i)} \leq z_j^{(i)}$. We then replace the $\min(z_j^{(i)}, x_j)$ terms in the objective with $\mu_j^{(i)}$. Since the linear programming is maximizing the $\mu_j^{(i)}$, at the optimal solution, each $\mu_j^{(i)}$ will be equal to $\min(z_j^{(i)}, x_j)$. Therefore, we have the linear program

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} \sum_{j=1}^{M} \mu_j^{(i)} \\ \text{subject to} \quad & \sum_{j=1}^{M} x_j = B \\ & x_j \leq c_j \quad (\forall j) \\ & \mu_j^{(i)} \leq x_j \quad (\forall i, j) \\ & \mu_j^{(i)} \leq z_j^{(i)} \quad (\forall i, j) \\ & x_j \geq 0 \quad (\forall j) \\ & \mu_j^{(i)} \geq 0 \quad (\forall i, j) \end{aligned}$$

This is polynomial-sized linear program with $MN + M$ decision variables and $2MN + M + 1$ constraints (excluding non-negativity constraints). The dual of this linear program will be significant in future lectures.

4.6 Problems with Knapsack Voting

Although the knapsack voting framework described the in previous sections has the advantage of being both strategy-proof and having low computational demands, it faces several issues that are not apparent in the mathematical formulation. One issue in particular is voter engagement with the participatory budgeting scheme. Those with low civic engagement may not vote, or in other cases a group of people may mobilize over a particular issue. Another issue is that the expense items may still be decided by bureaucrats, and may not be considered salient issues by voters.

Still a further issue is that a minority of voters may strongly value some project that is opposed by the majority. These voters might argue that a more fair result could be obtained if part of the budget proportional to the size of the majority is split off and given to the minority to manage. We might hope that our social function produces a budget that avoids this complaint. Such budgets are said to be in the "core," an idea central to a mathematical definition of fairness. For an in-depth discussion, see Fain et al. [2016] and Peters et al..

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