MS&E 336/CS 366: Computational Social Choice. Winter 2019-20
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7 From ordinal to cardinal social choice (cont'd)

When we have cardinal measure of social cost/utility but we only solicit ordinal preferences from individuals, the gap between the social cost/utility of the alternative chosen by the social choice function using only the ordinal preferences and that of the optimal alternative with known cardinal measure is called **distortion**.

Let u_{ij} be the utility of voter i for candidate j, and c_{ij} be the cost of votor i for candidate j.

7.1 Distortion

7.1.1 Utility model

Under the utility model, the distortion of candidate C_j is defined as:

$$D(C_j) = \frac{\max\limits_{\text{candidate } C_k} \sum\limits_{\text{voter } V_i} u_{ik}}{\sum\limits_{\text{voter } V_i} u_{ij}}$$

With randomized algorithm, we use $\mathbf{E}[u_{ij}]$, the expected utility of the chosen candidate for all voters, as the denominator.

7.1.2 Cost model

Under the cost model, the distortion of candidate C_j is defined as:

$$D(C_j) = \frac{\sum\limits_{\text{voter } V_i} c_{ij}}{\min\limits_{\text{candidate } C_k} \sum\limits_{\text{voter } V_i} c_{ik}}$$

With randomized algorithm, we use $\mathbf{E}[c_{ij}]$, the expected cost of the chosen candidate for all voters, as the numerator.

7.2 Reasonable restrictions

Given the impossibility results for both the utility and the cost model without restrictions, we impose the following reasonable restrictions for the following sections:

- **Utility model:** $\forall i, \sum_j u_{ij} = 1. \forall i, j, u_{ij} \geq 0$.
- **Cost model:** There is a metric space on voters and candidates s.t. $c_{i,j} = d(V_i, C_j)$.

7.3 Metric distortion under the utility model

Deterministic rule

Under the reasonable restrictions, the lower bound of the worst case distortion of a deterministic social choice rule is $\Omega(M)$.

Proof: Suppose there are N voters and M candidates, $N = M$, and the voters' preferences are as follows:

$$\begin{array}{cccccc} V_1 & V_2 & \dots & V_i & \dots & V_M \\ \hline \boxed{C_1} & C_2 & \dots & C_i & \dots & C_M \\ \vdots & \vdots & & \vdots & & \vdots \end{array}$$

Without loss of generality, suppose the given social choice function picks candidate C_1 as the winner. In the worst case, the underlying metrics can be:

$$u_{i,j} = \begin{cases} \frac{1}{M}, & \forall i = 1 \\ 1, & i = j \neq 1 \\ 0, & \text{otherwise} \end{cases}$$

The optimal utility is then $1 + \frac{1}{M}$ while the allocation utility is $\frac{1}{M}$. Thus, distortion $> M$.

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Randomized rule

Under the reasonable restrictions, the lower bound of the worst case distortion of a randomized social choice rule is $\Omega(\sqrt{M})$.

Proof: Assume that M is a perfect square and $N = k\sqrt{M}$. Partitions the voters into $\sqrt{m}$ equal subsets, $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_{\sqrt{m}}$, each with size k. Construct the following example profile: every voters in set $\mathcal{V}_i$ put C_i in position 1, the rest are ranked arbitrarily. Now assumes without loss of generality that the given social choice rule picks candidate C_1 with probability at most $\frac{1}{\sqrt{M}}$. In the worst case, the underlying metrics can be:

$$u_{i,j} = \begin{cases} 1, & j = 1, i \in \mathcal{V}_1 \\ 0, & j \neq 1, i \in \mathcal{V}_1 \\ \frac{1}{M}, & \text{otherwise} \end{cases}$$

The optimal utility, which is achieved by picking candidate C_1, is thus $\text{OPT} = k \cdot 1 + k(\sqrt{M} - 1) \cdot \frac{1}{M} > k$. Since candidate C_1 is picked by the given social choice rule with probability at most $\frac{1}{\sqrt{M}}$, its expected utility is thus $\leq \frac{1}{\sqrt{M}} \cdot \text{OPT} + (1 - \frac{1}{\sqrt{M}}) \cdot \frac{N-k}{M} \leq \frac{1}{\sqrt{M}} \cdot \text{OPT} + \frac{1}{\sqrt{M}} \cdot \frac{N}{\sqrt{M}} \leq \frac{1}{\sqrt{M}} \cdot \text{OPT}$. Therefore, distortion $\geq \frac{\sqrt{M}}{2}$. ■

Remark 7.1 *Note that this bound is very tight. In fact the upper bound for randomized rules is shown to be $O(\sqrt{M} \cdot \log^* M)$. Read up to Theorem 3.3 of [1] for details. Here we show an example algorithm which has an distortion upper bound of $O(\sqrt{M} \log M)$.*

When candidate C_j is in position k in voter V_i's ranking, it holds that $u_{ij} \le \frac{1}{k}$. Recall that $H_m = \sum_{k=1}^{m} \frac{1}{k}$ is the m-th harmonic number and $H_m \le \ln(1+m)$. Now assumes $\text{score}(C_j) = \sum_i \frac{1}{k_{ij}}$ where k_{ij} is the position of candidate C_j in voter V_i's ranking. Consider the following randomized algorithm:

1. With probability $\frac{1}{2}$, pick candidate C_j with probability $\propto \text{score}(C_j)$, i.e. candidate C_j is picked with probability $\frac{\text{score}(C_j)}{N\cdot H_M}$
2. With probability $\frac{1}{2}$, pick a candidate uniformly at random.

Analysis

Assume that C_{j*} is the best candidate. The optimal utility $u_{\text{opt}} = \sum_i u_{ij*}$. Consider the two cases where $u_{\text{opt}} \ge N \cdot \sqrt{\frac{H_M}{M}}$ and $u_{\text{opt}} < N \cdot \sqrt{\frac{H_M}{M}}$.

1. Suppose $u_{\text{opt}} \ge N \cdot \sqrt{\frac{H_M}{M}}$, then $\mathbf{Pr}[j^* \text{ gets picked}] \ge \frac{1}{2} \cdot \frac{\text{score}(C_{j*})}{N\cdot H_M} \ge \frac{1}{2}\frac{N\sqrt{\frac{H_M}{M}}}{N\cdot H_M} \ge \frac{1}{2\sqrt{M\cdot H_M}}$. Therefore distortion $\le 2\sqrt{M \cdot H_M}$.
2. Suppose $u_{\text{opt}} < N\cdot\sqrt{\frac{H_M}{M}}$, then $\mathbf{Pr}[j^* \text{ gets picked}] \ge \frac{1}{2}N\frac{1}{M}$, distortion $\le \frac{N\sqrt{\frac{H_M}{M}}}{\frac{1}{2}N\cdot\frac{1}{M}} \le 2\sqrt{M \cdot H_M}$.

We conclude that distortion $\le 2\sqrt{M \cdot H_M} \le 2\sqrt{M \ln(M+1)}$.

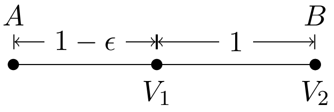
7.4 Metric distortion under the cost model

7.4.1 Distortion lower bound for deterministic social choice function

We begin by showing that the lower bound of worst-case distortion with metric costs is 3 using a simple example. Suppose there are two voters V_1, V_2 and two candidates A, B, and the profile is:

V_1	V_2
$\boxed{\text{A}}$	B
B	A

Without loss of generality, suppose A is chosen as the winner. The underlying metric space can be a line where $d(V_1, A) = 1-\epsilon, d(V_1, B) = 1, d(V_2, A) = 2-\epsilon, d(V_2, B) = 0$:



$D(A) = \frac{3-2\epsilon}{1}$. Thus, the distortion approaches 3 as $\epsilon \to 0$.

7.4.2 Distortion upper bound for Copeland social choice function

Copeland social choice function is known to have a distortion of at most 5[2]. Here we present an easy proof that Copeland has a distortion of at least 9.

PROOF OUTLINE: Suppose C is the Copeland winner. Without making any assumption of the metric space, we can show that:

1. For any C', either C beats C' in a pairwise election, or $\exists C''$ s.t. C beats C'' and C'' beats C' in a pairwise election.
2. If A beats B in a pairwise election, then $\sum\limits_{\text{Voters } i} d(i,A) \leq 3 \sum\limits_{\text{Voters } i} d(i,B)$.
3. With 1 and 2, it follows that for Copeland winner C and any candidate C', $\sum_i d(i,C) \leq 9 \sum_i d(i,C'')$, therefore $D(C)$ is at most 9.

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Proof:

1. We assume an odd number of voters, so there is no tie. Let $\mathbf{S}$ be the set of candidates that C beats in a pairwise election. When $C' \notin \mathbf{S}$, and there is no $C'' \in \mathbf{S}$ that C'' beats C', C cannot be the Copeland winner since C' beats C and all candidates in $\mathbf{S}$. Therefore, either C beats C' or $\exists C''$ s.t. C beats C'' and C'' beats C'.
2. Let V_1 be the set of voters that prefers A over B and V_2 be the set of voters that prefer B over A. If A beats B in a pairwise election, $|V_2| \leq |V_1|$. Therefore we can establish a matching $m(i) \in V_1$ for all $i \in V_2$. By triangle inequality,

$$\begin{aligned}
\sum_i d(i,A) &= \sum_{i \in V_1} d(i,A) + \sum_{i \in V_2} d(i,A) \\
&\leq \sum_{i \in V_1} d(i,B) + \sum_{i \in V_2} d(i,B) + \sum_{i \in V_2} d(m(i),B) + \sum_{i \in V_2} d(m(i),A) \\
&\leq \sum_{\text{Voters } i} d(i,B) + \sum_{i \in V_1} d(i,B) + \sum_{i \in V_1} d(i,A) \\
&\leq 3 \sum_{\text{Voters } i} d(i,B)
\end{aligned}$$

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References

[1] C. Boutilier et al. *Optimal social choice functions: A utilitarian view.* Artificial Intelligence 227 (2015) 190–213.

[2] K. Munagala et al. *Improved Metric Distortion for Deterministic Social Choice Rules.* EC 2019.