MS&E 336/CS 366: Computational Social Choice. Winter 2019-20
Course URL: `http://www.stanford.edu/~ashishg/msande336.html`.
Instructor: Ashish Goel, Stanford University.

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Introduction to game-theoretic concepts

We can think of social choice in game-theoretic terms: in a mapping of agents' actions to payoffs, we might think of voters' ballots as corresponding to actions, and the position of the output of the social choice function with respect to these ballots as corresponding to payoffs. Thus, it's useful in our study of social choice to first be introduced to some tools from game theory.

Strategy-proofness

In a multi-player game, a player A has a **dominant strategy** if there exists an action S such that for every other action S', playing S' can not be better for A than playing S regardless of what other players do, and there exists at least one scenario in which playing S is better.

Example 2.1 (Rock paper scissors (dominant strategy)) *In rock paper scissors with two players, the sets of possible actions for each player are:*

$$A : \{\text{rock}, \text{paper}, \text{scissors}\}$$
$$B : \{\text{rock}, \text{paper}, \text{scissors}\}$$

with the following payoff matrix for player A (with an analogous matrix for player B):

		Player A		
		Rock	Paper	Scissors
Player B	Rock	0	1	-1
	Paper	-1	0	1
	Scissors	1	-1	0

A dominant strategy does not exist for either player in this setup.

A mechanism is **incentive-compatible** if truthful reporting is a dominant strategy.[1] We demonstrate this idea in the following model of voting on a line.

[1] Same concept as strategy-proofness.

Voting on a line

Consider a model where each voter i has a most preferred point $x_i \in [0,1]$ (e.g., preferred size of a budget deficit) and utility $U_i(x)$ that is strictly decreasing with the distance from x_i (not necessarily symmetrically or smoothly). See Figure 1 for an example. Voter preferences in this setup are said to be **single-peaked**.

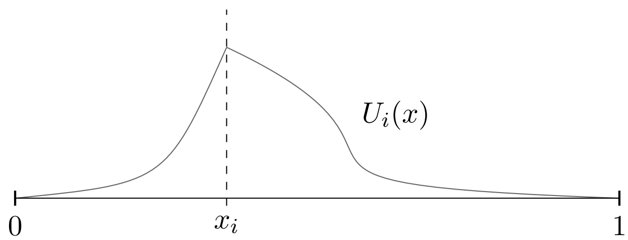


Figure 1: Utility $U_i(x)$ for voter i given x_i

An **elicitation rule** describes how preferences are elicited from voters (e.g., asking voter i for x_i). An **aggregation rule** describes how preferences are aggregated (e.g., taking the mean of x_i over all i). An elicitation rule and an aggregation rule together comprise a **voting mechanism**.

Using the mean as an aggregation rule is not incentive-compatible. Consider an example with four voters, where $x_1 = x_2 = x_3 = 0$ and $x_4 = 0.25$. If everyone reports truthfully, taking the mean yields 0.0625. On the other hand, if voter 4 reports $x_4' = 1$, aggregating via the mean yields 0.25, so voter 4 has an incentive to misreport.

Now suppose we choose for our aggregation rule the median[2] of x_i over all i.

Figure 2: Median as aggregation rule

Consider Figure 2. Suppose voter α's true preference is x_α, but they report x_α'. If the median is in the set of points denoted in red, this misreporting is inconsequential; if we think of the median as partitioning the set of x_is in two, then x_α and x_α' fall in the same "partition" with respect to the median so misreporting will not change the outcome.

On the other hand, suppose the median falls in the set of points denoted by the dashed line. Then misreporting x_α' will change the median, but in a way that moves the median away from x_α.

Since misreporting would be either inconsequential or result in a worse outcome, using the median as the aggregation rule is incentive-compatible (**Black's theorem**).[3]

[2] As an aside, Sir Francis Galton famously obtained a near-perfect estimate of the weight of an ox in 1907 by taking the median of estimates from 787 villagers.

[3] One application of Black's theorem is participatory budgeting.

Nash equilibria

Pure strategies

Given players 1 through n, a **pure strategy Nash equilibrium** is a set of actions $\{A_1, A_2, \ldots, A_n\}$ (where A_i denotes the action of player i) such that no player has an incentive to deviate.

A pure Nash equilibrium may not exist.

Example 2.2 (Rock paper scissors (pure strategy NE)) *Conditional on player* 1 *choosing to play rock, player* 2 *would profitably deviate by playing paper instead of, say, scissors. Conditional on player* 2 *playing paper, player* 1 *would profitably deviate by playing scissors instead of rock, etc., and so a pure strategy Nash equilibrium does not exist.*

Mixed strategies

A mixed strategy is one where players choose actions from some distribution. Players know each other's distributions but can't ex-ante know what particular action will be drawn. A **mixed strategy Nash equilibrium** is a set of distributions $\{D_1, D_2, \ldots, D_n\}$ such that no player can obtain a higher expected payoff by deviating.

Example 2.3 (Rock paper scissors (mixed strategy NE)) *Now suppose each player's strategy is to choose each action with equal probability, i.e.,* $\frac{1}{3}$. *This is a mixed strategy Nash equilibrium. No player can profitably deviate by choosing a different distribution across actions, e.g., if player* 1 *adjusts their strategy so that rock is played more often, player* 2 *can choose a strategy that plays paper more often.*

Pathological Nash equilibria in social choice

As seen in Examples 2.4 and 2.5, social choice contexts can exhibit pathological Nash equilibria.

Example 2.4 (Plurality) *Consider a setup with three voters* $\{v_1, v_2, v_3\}$, *three alternatives* $\{a, b, c\}$, *and winner determined by plurality. Suppose we have the truthful preferences presented in Table 1.*

v_1	v_2	v_3
a	b	c
b	c	a
c	a	b

Table 1: Pathological profile

Now suppose voters submit identical orderings (e.g., everybody reports $b \succ c \succ a$). No single voter can profitably deviate; a single voter's deviation will not change the outcome, as the winner is determined by plurality.

Example 2.5 (Copeland) *Suppose there are 101 voters, 52 of whom submit orderings that place the same alternative highest. No single voter can profitably deviate, as the outcome will not change even if they do.*

Note that in both cases, coalitions of voters can sidestep these equilibria.

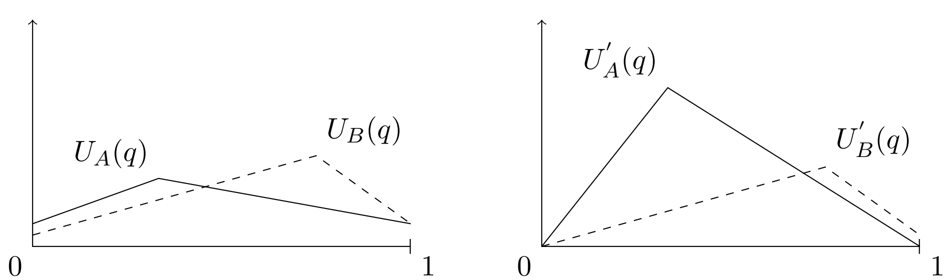
Bargaining

A general two-player bargaining problem can be set up as follows: players A and B negotiate given outside option Z. Let $S = \{(x_i, y_i)\}$ for all i denote the set of pairs of utilities such that if the two players agree on some outcome (x_A, x_B), then players A and B receive x_A and x_B, respectively. Otherwise, the outcome is the outside option (e.g., if $Z = (0, 0)$, then both players receive 0). Note that $Z \in S$.

Nash's axiomatic approach

Nash proposed an axiomatic approach to the bargaining problem, so that the only outcomes considered are those that satisfy (1) invariance to affine transformations, (2) Pareto optimality, (3) independence of irrelevant alternatives, and (4) symmetry. We describe each of these axioms with accompanying examples.

Consider two players A and B choosing among points in the interval $[0, 1]$. Suppose both players have utilities that are a function of q as in Figure 3a, i.e., $U_A(q^{'})$ is the utility player A obtains at point $q^{'} \in [0, 1]$. **Invariance to affine transformations** dictates that an affine transformation to the utilities will not change the bargaining outcome. That is, the outcome is invariant to adding/subtracting constants to the utilities or scaling them by a constant factor (as in Figure 3b). This also means that we can let $(0, 0)$ be the outside option without loss of generality.



(a) $U_A(q)$ and $U_B(q)$

(b) $U_A(q)$ and $U_B(q)$ affine transformed

Figure 3: Invariance to affine transformations

The red curve in Figure 4 traces out the Pareto frontier of utility payoffs for some bargaining problem. An outcome (U_A^*, U_B^*) is **Pareto-optimal** if no other outcome $(U_A^{'}, U_B^{'})$ results in a better payoff for both A and B. If an outcome is not on the Pareto frontier, then at least one player can achieve a better payoff without worsening the payoff of the other player. **Symmetry** dictates that if (x, y) is the outcome of the bargaining process and we switch the utility functions of the players, then (y, x) becomes the outcome.

Finally, let S denote the set of feasible outcomes and suppose the bargaining solution $(U_A^*, U_B^*) \in S$. **Independence of irrelevant alternatives** dictates that if $(U_A^*, U_B^*) \in S^{'}$ where $S^{'} \subseteq S$, then (U_A^*, U_B^*) is also the solution to the bargaining problem when $S^{'}$ is the set of feasible outcomes. The dashed curve in Figure 4 traces out the Pareto frontier for some $S^{'}$.

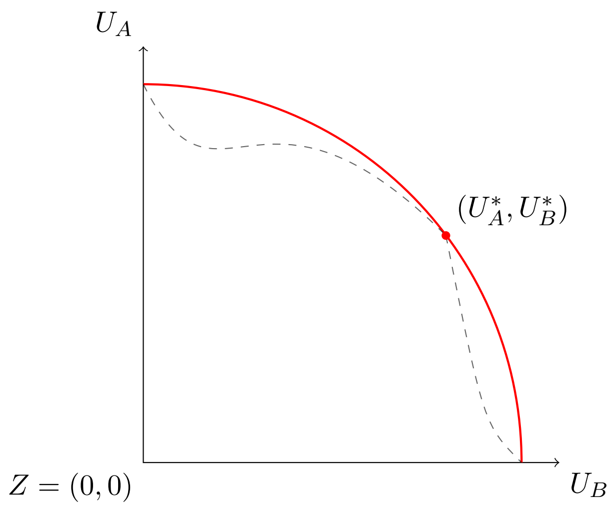


Figure 4: Pareto frontiers of payoffs

Example 2.6 (Independence of irrelevant alternatives) *Suppose an agent chooses burritos when faced with the decision of choosing between pizza, burritos, ramen, and burgers. Then faced with the decision of choosing between just burritos, ramen, and burgers, they should still choose burritos.*

Nash bargaining solution

With the outside option as $(0, 0)$ (recall that invariance to affine transformations allows this), the Nash bargaining solution is given by maximizing $U_A \cdot U_B$. Equivalently, the Nash bargaining solution is the solution to:

$$\max \quad \log(U_A) + \log(U_B)$$

subject to the constraint that the outcome must be feasible. This objective is also known as **Nash Welfare**.

Notice that multiplying the utilities by some constant will only add a constant term to the objective function, leaving the solution unchanged. In general, the Nash bargaining solution maximizes the product of the additional utilities each player would receive over the outside option.

The Nash bargaining solution is the unique solution that satisfies the four axioms. Note that this solution is obtained via assumption of the axioms, rather than being an equilibrium concept (e.g., based on player strategies).

Markets

Suppose we have a market with m divisible goods, n buyers, and a unit of money (say, a dollar) that has no value outside the market allocated to each buyer. Each good costs some amount of money, and buyer i has weight $w_{ij} > 0$ for good j so that buyer i's utility from buying x_{ij} amount of good j is $w_{ij}x_{ij}$.

Given these weights, we want to find prices that clear the market (i.e., every good is exactly sold). Each buyer i solves the following optimization problem:

$$\begin{aligned} \max_{x_{ij}} \quad & \sum_{\text{goods } j} w_{ij}x_{ij} \\ \text{subject to} \quad & \sum_j p_j x_{ij} \leq 1 \\ & w_{ij} > 0 \\ & x_{ij} \geq 0. \end{aligned}$$

We say that a set of prices clears the market if there is some solution to each buyer's optimization problem such that all the goods get exactly sold.

Note that supply constraints do not appear explicitly in the optimization (e.g., buyers don't care if Walmart runs out of toothpaste). Prices are chosen so that the market clears given each individual buyer's optimization problem. This setup is known as the **Fisher market**.

Example 2.7 (Fisher market prices) *Consider a market with set of buyers* $\{1, 2, 3\}$ *and set of goods* $\{A, B, C\}$ *and the following matrix of weights* w_{ij}*:*

		Good		
		A	B	C
Agent	1	2	2	1
	2	2	2	1
	3	2	2	1

Since every agent requires twice the amount of good C *as goods* A *or* B *to obtain the same utility, we would expect nobody to buy good* C *over good* A*, for example, if* $\text{price}_C > \frac{1}{2} \cdot \text{price}_A$*, as it would cost more to obtain the same utility.*

On the other hand, if $\text{price}_C < \frac{1}{2} \cdot \text{price}_A$ *and* $\text{price}_C < \frac{1}{2} \cdot \text{price}_B$*, we would expect all three agents to purchase good* C*. In the case where* $\text{price}_C = \frac{1}{2} \cdot \text{price}_A$ *and* $\text{price}_A = \text{price}_B$*, the three agents will be indifferent between the three goods.*

For example, if we had $\text{price}_A = \text{price}_B = \frac{6}{5}$*, and* $\text{price}_C = \frac{3}{5}$*, then the goods could be allocated among agents as follows:*

		Good		
		A	B	C
Agent	1	$\frac{2}{3}$	0	$\frac{1}{3}$
	2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	3	0	$\frac{2}{3}$	$\frac{1}{3}$

That is, buyer 1 *gets* $\frac{2}{3}$ *of good* A *and* $\frac{1}{3}$ *of good* C*, buyer* 2 *gets* $\frac{1}{3}$ *of each good, and buyer* 3 *gets* $\frac{2}{3}$ *of good* B *and* $\frac{1}{3}$ *of good* C*. In general, neither prices nor bundles need be unique.*

On the other hand, suppose we have the following matrix of weights instead:

		Good		
		A	B	C
Agent	1	1	1	2
	2	2	2	1
	3	2	2	1

With prices so that $\text{price}_A = \text{price}_B$ *and* $\frac{1}{2} \cdot \text{price}_A < \text{price}_C < \text{price}_A$*, agent* 1 *will purchase only good* C *while agents* 2 *and* 3 *will be indifferent between goods* A *and* B*. Using a solver, we obtain* $\text{price}_A = \text{price}_B = 0.434294164$ *and* $\text{price}_C = 0.434294075$*, with the following allocation:*

		Good		
		A	B	C
Agent	1	0	0	1
	2	0.5	0.5	0
	3	0.5	0.5	0

In general, agent i will buy the good j given by $\arg\max_j \frac{w_{ij}}{p_j}$.

The allocation obtained from clearing the Fisher market is invariant to buyers scaling their utilities by some constant factor, Pareto-optimal, symmetric, and anonymous. It is also **envy-free**, which is to say that no agent strictly prefers a bundle or good that someone else has.

Example 2.8 (Envy-freeness) *Consider the problem of allocating indivisible ice cream cones where agents have the following preferences over flavors:*

If Agent 2 receives vanilla and Agent 1 receives strawberry, we say that such an allocation ***induces envy****.*

Agent 1	Agent 2	⋯
Vanilla	**Vanilla**	⋯
Strawberry	Chocolate	⋯
Chocolate	Strawberry	⋯

The Fisher market allocation is envy-free in that each agent could have purchased any other agent's bundle, so the fact that they didn't implies that they do not strictly prefer that bundle to their own.

Example 2.9 (Pareto-optimal $\not\Rightarrow$ envy-free) *Suppose we have the following preferences:*

Agent 1	Agent 2
Vanilla	**Vanilla**
Strawberry	Strawberry

and allocation such that agent 2 receives vanilla while agent 1 receives strawberry.

This allocation is Pareto-optimal: we can only improve agent 1's payoff by giving them vanilla so that agent 2 receives a worse allocation of strawberry. However, it is not envy-free: agent 1 strictly prefers agent 2's allocation, vanilla, to their own.

The solution to the market-clearing problem with divisible goods is given by maximizing Nash welfare[4] (i.e., the product—equivalently, sum of logs—of the individual utilities) as in the following optimization problem:

$$\begin{aligned} \max_{x_{ij}} \quad & \sum_{\text{agents } i} \log\left(\sum_{\text{goods } j} w_{ij} x_{ij}\right) \\ \text{subject to} \quad & \sum_{\text{agents } i} x_{ij} \leq 1 \quad \forall j \\ & x_{ij} \geq 0. \end{aligned}$$

Note that $w_{ij} > 0$ for all i, j by assumption. This problem is also known as the **Eisenberg-Gale program**.

Taking the log of individual utilities is nice in that the allocation obtained has good fairness properties in the following sense: buyers with higher utilities for a good do not "monopolize" the purchase of that good. That is, allocating a fixed amount of good G to someone with a smaller utility for G will improve the value of the objective moreso than allocating that same amount to someone with a larger utility for G.

[4]Note that, like the Nash bargaining solution, the market-clearing solution fulfills Nash's bargaining axioms and is obtained by maximizing Nash welfare.

The allocation given by the solution to the Eisenberg-Gale program is not strategy-proof[5] and not a Nash equilibrium. However, it is still a widely used equilibrium concept.

Rough proof sketch of the market-clearing solution

A full proof is beyond the scope of this class, and involves the use of Lagrange multipliers and KKT conditions.

We can derive prices via first-order conditions of the Eisenberg-Gale program: the market-clearing price of good j is the value of the optimal Lagrange multiplier on constraint $\sum_{\text{agents } i} x_{ij} \leq 1$. The first-order condition for buyer i, good j is

$$\frac{w_{ij}}{\sum_{\text{goods } j} w_{ij} x_{ij}} - p_j = 0.$$

If $p_j > \frac{w_{ij}}{\sum_{\text{goods } j} w_{ij} x_{ij}}$, we would expect buyer i to not purchase any amount of good j, i.e., $x_{ij} = 0$. If $x_{ij} > 0$, then $p_j = \frac{w_{ij}}{\sum_{\text{goods } j} w_{ij} x_{ij}}$.

[5] See `gale-eisenberg.xlsx` on the course website.