MS&E 336/CS 366: Computational Social Choice. Winter 2019-20
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Core of the Participatory Budgeting problem

Objective of this lecture is to find/comment on whether a solution to the participatory budgeting problem is in the core. The relaxed version of the participatory budgeting problem is defined in the subsection below-

Relaxed problem of Participatory Budgeting:

There are N agents and M projects, we need to find a solution x where-

$$\text{Maximize} \sum_{user\ \ i} U_i(x) \quad s.t.\ x \geq 0 \tag{1}$$

$$\sum_{projects\ \ j} x_j \leq B$$

We make the following further assumptions:

- We assume that there is no cap on the money that can be allocated to a single project.
- There are a small number of agents
- $U_i(x) = \sum_j u_{ij} x_j \quad (u \geq 0)$
- Budget $B = 1$.

Definition of Core is provided in the following subsection-

Definition of Core:

Consider a solution x, where x_j is the money allocated to project j and-

$$x \geq 0,\ \sum_j x_j \leq B \tag{2}$$

This solution is said to be in the core if there does not exist a subset of agents $S \subseteq \{1, 2, 3, 4, ... N\}$ and a solution z such that:

$$z \geq 0,\ \sum_j z_j \leq \frac{B \mid S \mid}{N} \tag{3}$$

and-

$$\forall\ i\epsilon S,\ \sum_j z_j u_{ij} > \sum_j x_j u_{ij} \tag{4}$$

If such a subset S were to exist, it would be called the **defecting coalition**.

A solution is said to be in the core if no subset of agents are happier (receive a better utility) if they were given their share of the budget and allowed to allocate it to projects however they liked.

Note that if a solution is in the core, then it satisfies the following-

- Pareto Optimality: The defecting coalition is the set of all agents.
- Proportionality: The defecting coalition is a subset containing a single agent.

Examples of solutions not in the core:

Example 1

Consider the following example with N agents where N is odd and 2 projects. The utilities of each user for each of the projects is shown in the table below-

Agent i	u_{i1}	u_{i2}
1	1	0
2	1	0
3	1	0
4	1	0
:	:	:
$\frac{N+1}{2}$	1	0
$\frac{N+1}{2}+1$	0	1
:	:	:
N	0	1

Consider the solution in which the majority voted project is selected. Then, agents $\frac{N+1}{2}+1$ to agents N belong to a Defecting coalition. So the solution according to the Majority rule is not in the core.

Example 2

Consider the solution to the maxmin rule - Max $\text{Min}_{\text{agent } i} U_i(x)$ for the following example-

Agent i	u_{i1}	u_{i2}
1	1	0
2	1	0
3	1	0
4	1	0
:	:	:
:	:	:
N	0	1

The solution according to the maxmin objective would be $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$. But we see that with this solution, agents 1 to $N-1$ are in the defecting coalition. So the solution from optimizing the maxmin objective is not in the core.

Example 3

Consider the following example with N agents and 3 projects. The utilities of each user for each of the projects is shown in the table below-

Agent i	u_{i1}	u_{i2}	u_{i3}
1	$\frac{3}{5}$	0	$\frac{2}{5}$
2	$\frac{3}{5}$	0	$\frac{2}{5}$
$\vdots$	$\frac{3}{5}\vdots$	$\vdots$	$\vdots$
$\frac{N}{2}$	$\frac{3}{5}$	0	$\frac{2}{5}$
$\frac{N}{2}+1$	0	$\frac{3}{5}$	$\frac{2}{5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	0	$\frac{3}{5}$	$\frac{2}{5}$

Consider the solution in which $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$. While this intuitively and proportionally might seem like the right solution, note that the total utility in this case is-

$$U = \frac{3}{5} \times \frac{N}{2} \times \frac{1}{2} \times 2 = \frac{3N}{10} \tag{5}$$

However instead if we look at the solution $x_3 = 1$, we notice that the total utility is-

$$U = \frac{2}{5}N \tag{6}$$

And we see that this is clearly higher than what is obtained earlier in equation (5).

Lindahl Equilibrium

Even though Lindahl Equilibrium is defined for the clearing of private and public markets, the same theory can be applied in this situation of participatory budgeting. In this situation, the notion of market clearing is achieved when all the agents "agree" on a solution. We will define personalized prices for each agent for each project $p_{ij} > 0$. For each agent i, there would be a solution y that is the solution to the optimization problem-

$$\text{Max}_y \sum_j u_{ij} y_j \tag{7}$$

$$s.t. \;\; y \geq 0 \;\; \& \; \sum_j p_{ij} y_j \leq \frac{B}{N} \tag{8}$$

A set of personalized prices p and a solution x together form a market clearing solution if the following are all simultaneously true:

1. x is Argmax of equations (5) for all agents.

2. x is Argmax for the centralized problem defined below-

$$\text{Max}_z \sum_i \sum_j p_{ij} z_j - \sum_j z_j \tag{9}$$

$$s.t. \quad z \geq 0 \tag{10}$$

Equation (7) acts as the "revenue" in this problem.

At equilibrium, the market clearing solution satisfies the following-

$$\sum_i \sum_j p_{ij} z_j - \sum_j z_j \leq 0 \tag{11}$$

As the revenue is a linear combination of the solution z, if the revenue was positive, then by doubling the value of z, the revenue will increase. So the solution would be $z = \infty$, but since z is also a solution to equation (7), we see that z has to take on finite values. Hence we have reached a contradiction. At the same time the revenue can also not be less than 0 as by setting $z = 0$, we can get the max value of $revenue = 0$. So the value of revenue should be exactly equal to 0 at equilibrium when the market clears.

$$\sum_i \sum_j p_{ij} z_j - \sum_j z_j = 0 \tag{12}$$

Claim: Suppose p, x is a Lindahl Equilibrium, then X is in the Core.

Proof:

Suppose x is not in the core. Then,

1. $\exists\, S, y$ such that they form a defecting coalition.
2. $\forall$ agent i ϵ S,

$$\sum_j p_{ij} y_j > \frac{B}{N} \tag{13}$$

Equation (11) is true because the fact that they bought x over y implies that even though y makes them happier, it is more expensive than x. Since y makes agent i happier than the equilibrium solution, it must violate equation (8), since otherwise the equilibrium solution could not have been a maximizer for the problem of optimization of agents utilities.

At equilibrium,

$$f(x) = \sum_i \sum_j p_{ij} x_j - \sum_j x_j = 0 \tag{14}$$

$$f(y) = \sum_i \sum_j p_{ij} y_j - \sum_j y_j \geq \sum_{i \epsilon S} \sum_j p_{ij} y_j - \sum_j y_j \tag{15}$$

$$\sum_{i \epsilon S} \sum_j p_{ij} y_j - \sum_j y_j > \frac{\mid S \mid B}{N} - \sum_j y_j > 0 \tag{16}$$

Since, $\sum_j y_j \leq \frac{|S|B}{N}$. But if $f(y) > 0$ then x could not have been the optimizer for the centralized problem, and hence p, x could not have been an equilibrium. So we have arrived at a contradiction.

Existence of the Lindahl Equilibrium

Technical Lemma: An allocation $x \geq 0$ corresponds to a Lindahl Equilibrium for the Participatory Budgeting problem if and only if for all items j-

$$\sum_j \frac{u_{ij}}{\sum_k u_{ik} x_k} \leq \frac{N}{B} \tag{17}$$

And this inequality is tight if $x_j > 0$.

We need to show that the solution of

$$\text{Max} \sum_i \log(\sum_j u_{ij} x_j) \tag{18}$$

subject to $\sum_j x_j \leq B,\ x \geq 0$ satisfies the technical lemma. We use the KKT theorem to find the solution to equation (16).

$$L(x) = \sum_i \log(\sum_j u_{ij} x_j) - \lambda(\sum_j x_j - B) \tag{19}$$

$$\frac{\partial L(x)}{\partial x_j} = \sum_j \frac{u_{ij}}{\sum_k u_{ik} x_k} - \lambda \tag{20}$$

If $x_j > 0$, $\frac{\partial L(x)}{\partial x_j} = 0$ at optimum. If $x_j = 0$, $\frac{\partial L(x)}{\partial x_j} \leq 0$. We claim that $\sum_j x_j \frac{\partial L(x)}{\partial x_j} = 0$. This implies-

$$\sum_i \sum_j \left[\frac{u_{ij} x_j}{\sum_k u_{ik} x_k} \right] - \lambda \sum_j x_j = 0 \tag{21}$$

$$\sum_j \frac{u_{ij} x_j}{\sum_k u_{ik} x_k} = 1 \tag{22}$$

This implies-

$$N - \lambda B = 0 \Rightarrow \lambda = \frac{B}{N} \tag{23}$$