MS&E 336/CS 366: Computational Social Choice. Aut 2010-21
Course URL: `http://www.stanford.edu/~ashishg/msande336/index.html`.
Instructor: Ashish Goel, Stanford University.

Lecture 8, 10/18/2021. Scribed by Andy Wang.

1 Recap

Given two candidates a, b, $a \geq_i b$, means that candidate a was ranked higher than candidate b by voter i. This means that for metric, d, $d(i,a) \leq d(i,b)$. Distortion, D, of candidate c is then defined as:

$$D(c) = \frac{\sum_v d(v,c)}{\min\limits_{c'} \sum_v d(v,c')} \tag{1}$$

This equation is difficult because d is unknown. Running an algorithm to choose candidate c using distortion D results in the equation:

$$D(c_{alg}) = \max_d \frac{\sum_v d(v,c_{alg})}{\min\limits_{c'} \sum_v d(v,c')} \tag{2}$$

This equation can be understood as the total unhappiness with a chosen candidate vs the minimum total unhappiness with any candidate. Since the metric is unknown, we take the maximum over all metrics consistent with the profile. From the previous lecture it was seen that the lower bound of worst-case distortion with metric costs was 3.

2 Distortion with Metric Costs

1. Copeland Proof

Copeland social choice is known to give a distortion of at most 5 [1]. Here we will give an easy proof of distortion of at most 9. To prove a bound of 9, we first show two statements are true.

1. if C is the Copeland winner, then for any other candidate C', either C beats C' in a pairwise election, or there exists a C" such that C beats C" and C" beats C'.

   Proof: Proof by contradiction: Given C is our copeland winner we have that

   (a) $C' >_{Copeland} C$ [Copeland: More voters prefer C' to C]

   (b) $\forall C''$ s.t. $C >_{copeland} C''$ then $C' >_{copeland} C''$

   This means that C' wins more pairwise elections than C. Thus, C cannot be the copeland winner. Meaning we have a contradiction. ■

2. If C beats C' in a pairwise election, then C is at most a factor of 3 worse than C'.

Claim 2.1 $\sum_v d(v,C) \leq 3\sum_v d(v,C')$.

Proof: Given a set of voters, V, we have that $S = \sum_{v \in V} v : C \geq C'$. Intuitively, S can be defined as the set of voters that prefers candidate C over C'. We also know that $|V| = N$ where N is the number of voters and odd. If C beats C' in a pairwise election, $|S| \geq \frac{N}{2}$ and $|S| \geq |V - S|$.

Thus, we can define two spaces S and $V - S$.
For every voter $w \in V - S$. We assign a unique node $R(w) \in S$. Note R(w) maps into a value in $V - S$. By Triangle Inequality:

$$\sum_{v \in V} d(v,C) = \sum_{v \in S} d(v,C) + \sum_{v \in V-S} d(v,C) \tag{3}$$

$$\sum_{v \in V-S} d(v,C) \leq \sum_{v \in V-S} d(v,C') + d(R(v),C) + d(R(v),C') \tag{4}$$

$$\sum_{v \in V} d(v,C) \leq \sum_{v \in S} d(v,C) + \sum_{v \in V-S} d(v,C') + d(R(v),C) + d(R(v),C') \tag{5}$$

Since $d(R(v),C) \leq d(R(v),C')$ for $v \in V - S$. This implies that

$$\sum_{v \in V-S} d(v,C') + d(R(v),C) + d(R(v),C') \leq \sum_{v \in V-S} d(v,C') + 2d(R(v),C') \tag{6}$$

Because more voters prefer C to C' we see the following is true:

$$\sum_{v \in S} d(v,C) \leq \sum_{v \in S} d(v,C') \tag{7}$$

If we combine equation 6 and 7 we get.

$$\sum_{v \in V} d(v,C) \leq \sum_{V \in S} d(v,C') + \sum_{V \in V-S} d(v,C') + 2d(R(v),C'). \tag{8}$$

Since $|V - S| < |S|$, we can say

$$\sum_{v \in V-S} d(R(v),C') < \sum_{v \in S} d(v,C') \tag{9}$$

Thus equation 8 can be converted to,

$$\sum_{V \in S} d(v,C') + \sum_{V \in V-S} d(v,C') + 2d(R(v),C') \leq 3\sum_{V \in S} d(v,C') + \sum_{V \in V-S} d(v,C'). \tag{10}$$

$$3\sum_{V \in S} d(v,C') + \sum_{V \in V-S} d(v,C') \leq 3\sum_{V \in S} d(v,C') + 3\sum_{V \in V-S} d(v,C') = 3\sum_{v} d(v,C') \tag{11}$$

Finally, collecting terms we get that $\sum_v d(v,C) \leq 3\sum_v d(v,C')$. Which is what we were trying to show. ■

3. By combining 1 and 2 we get that for a Copeland winner C and any candidate C' $\sum_v d(v, C) \leq 3 * 3 \sum_v d(v, C'')$. Thus D(C) has an upper bound of 9

2. Existence is Enough

We can show that as long as we commit to minimizing metric distortion, then we don't need to know the social choice rule which achieves it. Given the "optimum" candidate C', we can compute the distortion of any candidate C against C'. This is defined as:

$$D(C, C') = \max_d \frac{\sum_v d(v, C)}{\sum_v d(v, C')} \tag{12}$$

Constraints:

- $\forall v, x : d(v, x) \geq 0$
- $\forall v : d(v, v) = 0$
- $\forall x : d(x, x) = 0$
- $\forall v, x : d(v, x) = d(x, v)$
- $\forall v_1, v_2, x_1, x_2 : d(v_1, x_1) \leq d(v_1, x_2) + d(v_2, x_1) \leq d(v_2, x_2)$

From the original equation, we remove $\sum_v d(v, C)$ and add it as a constraint. We can then represent distortion, D, for a candidate, C, as $D(C) = \max_{C'} D(C, C')$. Thus, the optimal choice is the candidate C that minimizes this D(C), $\min_c D(C)$.

References

[1] Elliot Anshelevich, Onkar Bhardwaj, Edith Elkind, John Postl, and Piotr Skowron. Approximating optimal social choice under metric preferences. Artificial Intelligence, 264:27–51, 2018