MS&E 336/CS 366: Computational Social Choice. Aut 2021-2022
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8 Participatory Budgeting

Participatory budgeting (PB) is a process by which communities collectively decide on the allocation of public tax dollars for local public projects. First developed in Brazil in the 1980s, PB has become widespread in Brazilian cities and has been adopted in cities around the world [3].

In the standard formulation of the problem, we would like to devise a method for collecting voter's preferences for budget allocation into an aggregate budget which maximizes some global notion of utility for voters. Formally, we set up the problem as follows:

- Let N be the number of voters, M the number of projects or budget items, and B the total budget to be allocated.
- Let $\mathbf{x}^{(i)} = \langle x_1^{(i)}, \ldots, x_M^{(i)} \rangle$ be the ideal budget for voter i, where we suppose $x_j^{(i)}$ is voter i's **true** preference for number of dollars spent on project j.
- We assume all $\mathbf{x}^{(i)}$ are correct budget allocations. In other words, $\|\mathbf{x}^{(i)}\|_1 = B$ for all i and $x_j^{(i)} \geq 0$ for all i, j.
- Let $\mathbf{z} = \langle z_1, \ldots, z_M \rangle$ be the aggregated budget.

We define the *overlap utility* of an aggregated budget $\mathbf{z}$ for voter i as $U_i(\mathbf{z}) = \sum\limits_{j=1}^{M} \min\{z_j, x_j^{(i)}\}$.

Remark 8.1 *Overlap utility yields an equivalent notion of cost* $d_i(\mathbf{z}) = |\mathbf{x}^{(i)} - \mathbf{z}\|_1$

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Proof: Note that, for any i, $\sum\limits_j x_j^{(i)} = \sum\limits_j z_j = B$. Reshuffling the terms, we have that

$\sum\limits_{j:\, x_j^{(i)} \leq z_j} (z_j^{(i)} - x_j) = \sum\limits_{j:\, x_j^{(i)} > z_j} (x_j^{(i)} - z_j)$. Notice that the summation terms on both sides are equal to the absolute value $|x_j^{(i)} - z_j|$. Thus

$$\begin{aligned}\sum_{j:\, x_j^{(i)} > z_j} (x_j^{(i)} - z_j) &= \frac{1}{2}\left[\sum_{j:\, x_j^{(i)} > z_j} |x_j^{(i)} - z_j| + \sum_{j:\, x_j^{(i)} \le z_j} |x_j^{(i)} - z_j|\right] \\ &= \frac{1}{2}\sum_j |x_j^{(i)} - z_j| \\ &= \frac{1}{2}\|\mathbf{x}^{(i)} - \mathbf{z}\|_1\end{aligned}$$

Finally, we see that

$$U_i(\mathbf{z}) = \sum_{i=1}^{M} \min\{z_j, x_j^{(i)}\} = \sum_{i=1}^{M} x_j^{(i)} - \sum_{j:\, x_k^{(i)} > z_j} (x_j^{(i)} - z_j) = B - \frac{1}{2}\|\mathbf{x}^{(i)} - \mathbf{z}\|_1 = B - \frac{1}{2} d_i(\mathbf{z})$$

■

Intuitively, overlap utility/cost corresponds to the number of dollars that need to be shifted from each project for the aggregate budget to match the voter's ideal budget.

If we are satisfied with overlap utility/cost as representative of voter happines, we can cast participatory budgeting as the following optimization problem:

$$\begin{aligned} \arg\max_{\mathbf{z}} \quad & \sum_{i=1}^{N}\sum_{j=1}^{M} \min\{z_j, x_j^{(i)}\} \\ \text{subject to} \quad & \sum_{j=1}^{M} z_j = B; z_j \ge 0 \text{ for all } j \end{aligned} \tag{1}$$

By introducting additional decision variables t_{ij} for $1 \le i \le N, 1 \le j \le M$, we can equivalently formulate (1) as a linear program (see: [8] for an introduction), lending the problem to solution using any standard linear optimization algorithm:

$$\begin{aligned} \arg\max_{t_{ij}, \mathbf{z}} \quad & \sum_{i=1}^{N}\sum_{j=1}^{M} t_{ij} \\ \text{subject to} \quad & t_{ij} \le x_j^{(i)};\ t_{ij} \le z_j \\ & \sum_{j=1}^{M} z_j = B;\ z_j \ge 0 \text{ for all } j \end{aligned} \tag{2}$$

Claim 8.1 (PB as a linear program) *Suppose* $(\mathbf{t}, \mathbf{z})$ *is a solution to (2). Then* $\mathbf{z}$ *is a solution to (1).*

PROOF OUTLINE: Since $\mathbf{t}$ is optimal, t_{ij} must be equal to $\min\{z_j, x_j^{(i)}\}$. Otherwise, we could increase t_{ij}, thereby increasing the objective sum, without violating any constraints. Thus, the corresponding optimal $\mathbf{z}$ must also maximize the sum in (1). ■

Formulated in this manner, participatory budgeting is an example of a Knapsack problem. One method for practically solving the problem is to elicit voters to select any number of projects which satisfy budget constraints. Consequently, projects are ranked by order of number of voters, and the top projects are selected until the budget is exhausted. As shown in [5], such a method is strategy-proof and welfare maximizing under the cost model.

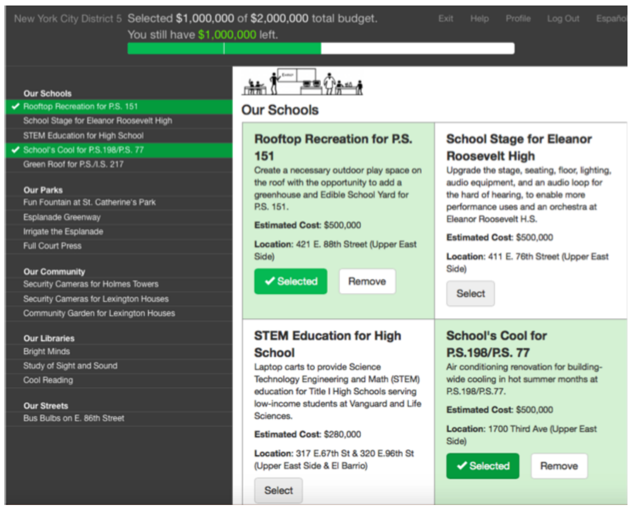


Figure 1: Example interface for Knapsack Voting developed in [5]

9 From Ordinal to Cardinal Social Choice

The social choice functions we have seen in class - and the vast majority of social choice mechanisms in place in contemporary democracies - all involve aggregation over ordinal preferences of voters. However, in evaluating the quality of a social choice function, it can be helpful to assume that voters implicitly assign costs or utilities to the different alternatives which induce their ordinal preferences. Assuming such cardinal utilities/costs allows us to evaluate an SCF based on its **distortion**, or the worst-case ratio of the utility/cost of the candidate selected by the function to the optimal alternative [7].

9.1 Distortion Bounds

Suppose we have voters $v_1, \ldots, v_N$, and candidates $c_1, \ldots, c_M$. We can consider distortion based on a utility or cost model of voter preferences. We assume that the designer of the SCF has no access to cost/utility information.

Under a utility model with utility function U, the distortion for candidate c_j is defined as

$$D(c_j) = \frac{\max\limits_{j'} \sum\limits_{i=1}^{N} U_i(c_{j'})}{\sum\limits_{i=1}^{N} U_i(c_j)}$$

Under a cost model with cost function d, the distortion is defined as

$$D(c_j) = \frac{\sum_{i=1}^{N} d_i(c_j)}{\min\limits_{j'} \sum\limits_{i=1}^{N} d_i(c_{j'})}$$

9.1.1 Unrestricted cost/utility

If we impose no restrictions on the structure of costs/utilities for voters, we encounter so-called impossibility results for bounding distortion, with the following lower bounds in the worst-case [7]:

	Deterministic SCF	Randomized SCF
Utility	∞	M
Cost	∞	∞

Example 9.1 (Unbounded disortion) *Consider the following profile:*

	U_1	U_2
A	0	1
B	0	0

We see that the distortion in case the SCF picks B is equal to $D(B) = \frac{1+0}{0} = \infty$.

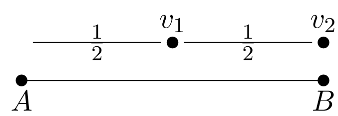
9.1.2 Utility restrictions

We impose the restriction that all voters have utilities over candidates which sum to 1, i.e. $\sum\limits_{j} U_i(c_j) = 1$ for all i. In this case, a determinstic SCF has a distortion lower bound of M while a randomized SCF has lower and upper bounds of $O(\sqrt{M})$ [2].

9.1.3 Cost restrictions

We assume that costs are distances in a metric space on the set of voters and candidates. A series of results has shown determinstic SCFs to have a tight upper bound of 3. For randomized SCFs, there is a known upper bound of 3 and tight lower bound of 2 [1][4][6].

Example 9.2 (Distortion under metric cost) *Suppose we have two voters v_1, v_2 and candidates A and B. Suppose that $d(v_1, A) = d(v_1, B) = \frac{1}{2}$ and $d(v_2, A) = 1, d(v_2, B) = 0$. The optimal cost of $\frac{1}{2}$ is obtained by picking candidate B. The worst-case cost of $\frac{3}{2}$ is obtained by picking candidate A. The distortion in this case achieves the upper bound of 3.*



Voter distance to candidates

References

[1] Elliot Anshelevich, Onkar Bhardwaj, Edith Elkind, John Postl, and Piotr Skowron. Approximating optimal social choice under metric preferences. *Artificial Intelligence*, 264:27–51, 2018.

[2] Craig Boutilier, Ioannis Caragiannis, Simi Haber, Tyler Lu, Ariel D Procaccia, and Or Sheffet. Optimal social choice functions: A utilitarian view. *Artificial Intelligence*, 227:190–213, 2015.

[3] Brandon Fain, Ashish Goel, and Kamesh Munagala. The core of the participatory budgeting problem. In Yang Cai and Adrian Vetta, editors, *Web and Internet Economics*, pages 384–399, Berlin, Heidelberg, 2016. Springer Berlin Heidelberg.

[4] Vasilis Gkatzelis, Daniel Halpern, and Nisarg Shah. Resolving the optimal metric distortion conjecture. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1427–1438. IEEE, 2020.

[5] Ashish Goel, Anilesh K Krishnaswamy, Sukolsak Sakshuwong, and Tanja Aitamurto. Knapsack voting for participatory budgeting. *ACM Transactions on Economics and Computation (TEAC)*, 7(2):1–27, 2019.

[6] Kamesh Munagala and Kangning Wang. Improved metric distortion for deterministic social choice rules. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 245–262, 2019.

[7] Ariel D. Procaccia and Jeffrey S. Rosenschein. The distortion of cardinal preferences in voting. In Matthias Klusch, Michael Rovatsos, and Terry R. Payne, editors, *Cooperative Information Agents X*, pages 317–331, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.

[8] Paul R Thie and Gerard E Keough. *An introduction to linear programming and game theory.* John Wiley & Sons, 2011.