MS&E 336/CS 366: Computational Social Choice. Aut 2010-21
Course URL: `http://www.stanford.edu/~ashishg/msande336/index.html`.
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Lecture 12, 10/27/2021. Scribed by Bing Liu.

13 Fisher Market c'td

In the last lecture (refer to previous notes for specifics.), we defined the market clearing conditions for Fisher Markets and also stated a theorem which states that

1. market clearing exists
2. the solutions in terms of prices and allocations can be found by solving the Gale Eisenberg convex program.

In this lecture, we prove the theorem and illustrate the solutions with some examples (one can play with the excel file sent in the class email).

1. Proof sketch

Proof sketch:

- rewriting the problem, let (the Lagrangian)

$$\max_{x_{ij}} L = \log(\sum_j u_{ij}x_{ij}) - \sum_j \lambda_j(\sum_i x_{ij} - 1) \tag{1}$$

$$\text{s.t.} \lambda_j(\sum_i x_{ij} - 1) = 0 \forall j \tag{2}$$

- take the first derivative w.t. x_{ij}:

$$\Delta_{ij} \equiv \frac{\partial L}{\partial x_{ij}} = \frac{u_{ij}}{\sum_k u_{ik}x_{ik}} - \lambda_j \tag{3}$$

  then (by optimality of x) either 1) $x_{ij} = 0$ and $\Delta_{ij} \leq 0$ (o/w we can increase x_{ij} to increase L.) or $\Delta_{ij} = 0$ (saddle point).

- Focusing on a any non-zero x_{ij} and $x_{ij'}$ (that is any two goods the agent bought a positive amount of), we have

$$\frac{u_{ij}}{\lambda_j} = \frac{u_{ij'}}{\lambda'_j} \tag{4}$$

  then market clearing prices are λ up to scaling (such that budget constraints are satisfied), and suggesting that agents are maximizing their utilities.

2. Illustration

We filled in some arbitrary valuations from the agents for the goods and computed the market clearing allocations and prices. Here are a few observations

1. how the agent allocates her budget on different good only depends on ratios of utilities, this is consistent with (4)

2. to compute the market clearing prices p, we can use the market clearing condition that $xp = \mathbf{1}$. Then $p = x^{-1}\mathbf{1}$.

3. market clearing prices are λ (the dual variables in Gale Eisenberg) up to scaling.

4. note the envy-free-ness of the allocation ('I' don't want what 'you' have). This is a result from the individual utility optimization.

Some clarifications about the ***market clearing*** condition

1. for the individual utility optimization condition in the market clearing condition, it is possible the agent wants to buy more than supply.

14 Public decision-making

The setup to the problem is that: there are multiple issues; for each issue, users have binary preferences ('for'(1) or 'against'(0) but the public decision z_l can be fractional; each user's utility is a weighted sum of its utility from each issue, where we use w_{il}: weights assigned to issue l by user i. Each user's utility of issue l given the public decision z_l is z_l if her preference is 'for' the decision; $1 - z_l$ otherwise.

Alluding to market-based solutions

'one person one vote'

Consider 'one person one vote' and majority voting on every issue. This approach may seem fair but there might be potential (large) welfare loss compared to social optimal solution.

Example 14.1 *Consider issues* $1, 2, 3$ *and agents* A, B, C *as illustrated in Fig 1. Using 'one person one vote' and majority voting on every issue, the decision is* 100% *on side* 1*. The final sum of utilities is* 6*. On the other hand, the decision of* 1 *on side 2 for all issues gives a sum of utilities of* 12*. This indicates a great welfare loss.*

The welfare loss is partly a result of the 'independentness' of decision making on different issues while users obtain weighted sum of utilities from all issues.

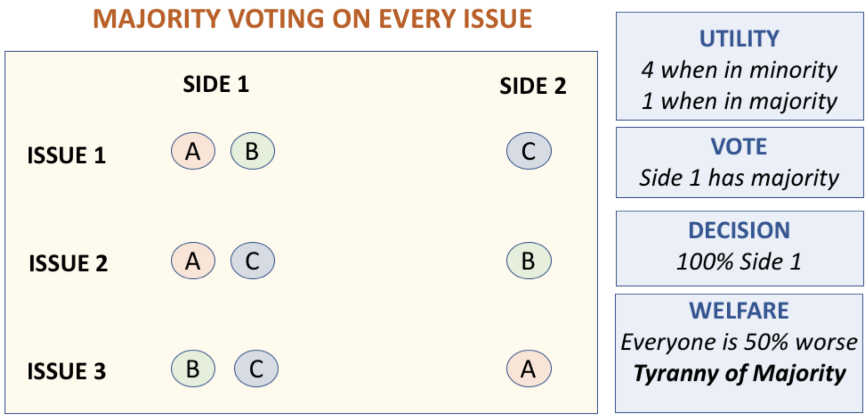

Figure 1: majority voting

Simple public market

Now let's consider a market approach where each issue has some price and agents can buy 'probabilities' on the issues (analogous to fractional allocation in the fisher market.). It seems that with prices and budget constraints, agents can be more 'expressive' of their valuations of the decisions on each issue, unlike what happened in 14.1. However, there is another layer of issue: the individual decisions are independent while the allocations are 'shared'. There may exists (positive) externalities from the allocation. Consider the following example

Example 14.2 *Consider the setup as illustrated by Fig 2. As the setup is completely symmetric, the market clearing price will be such that all the prices for the issues are identical. Then, the market clearing equilibrium is that agents A, B, C each put all their money on side* 2 *of issues* $3, 2, 1$ *respectively. (take C for example, the marginal utility of putting money on issue 2 or 3 is* 1 *while the marginal utility of putting money on issue 1 is* 1.1.*) Hence, the resulting welfare is* 3.3 *(from having* 100% *on side 2) while having* 100% *on side 1 yields a welfare of* 6.

Next class:

Modifying the market idea.

Interesting concepts to look up

- shapley-shubik market

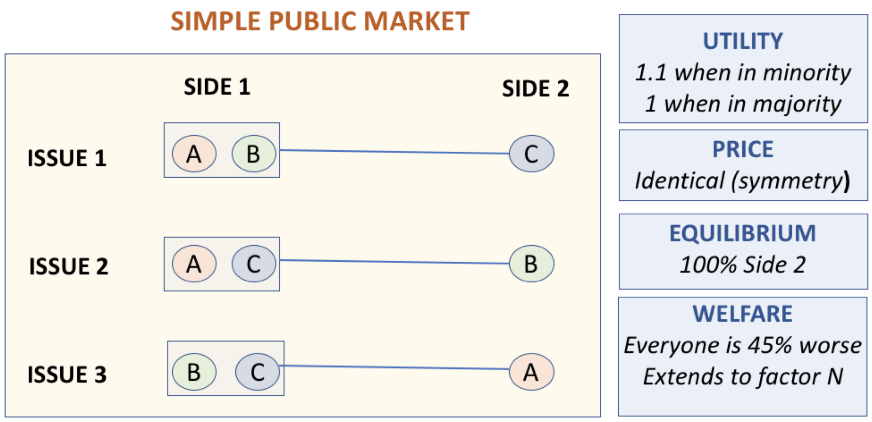

Figure 2: Simple public market