MS&E 336/CS 366: Computational Social Choice. Aut 2010-21
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12 Sequential Deliberation

Refer to Lecture 10 scribed notes for a detailed introduction to sequential deliberation and its associated results.

12.1 Some Results

This lecture involved showing two of the mentioned results.

Result 12.1 *We can analytically compute bounds on approximating the social cost minimizer by embedding onto the hypercube.*

Result 12.2 *The chosen alternative converges to a stationary distribution in O(1) steps*

12.1.1 Bounds on approximating the social cost minimizer

Assume N points on the line $x_1, ..., x_N$ If current solution is x_i, then next solution will be
(a) x_i, if we choose one point from each side of x_i
(b) at least x_j, where $j > i$, if we choose both points to be at least as large as x_j
(c) at most x_j, where $j < i$, if we choose both points to be at most as large as x_j

We can then define two quantities P_i and c_i.

- P_i is defined as the probability that x_i is the current solution in stationary distribution. This is similar to running N simulations and seeing when x_i is the solution.
- $c_i = \sum_{j=1}^{i} p_j$ This can be thought of as the probability of choosing some x_a s.t. $a <= i$

We can then compute c_i as follows.

$$c_i = c_i(1 - \frac{(N-1)^2}{N^2}) + (1 - c_i)\frac{i^2}{N^2}$$

The first term corresponds to the previous solution being in the same side (i.e. x_a s.t. $a <= i$) and not wanting to pick both points from the other side (i.e. x_a s.t. $a > i$). The second term corresponds to the previous solution being in the other side (i.e. x_a s.t. $a > i$) and wanting to pick both points from the same side (i.e. x_a s.t. $a <= i$). With some manipulation, we get the following

$$\frac{c_i(N-1)^2}{N^2} = (1-c_i)\frac{i^2}{N^2}$$

$$c_i = \frac{i^2}{i^2 + (N-1)^2}$$

Note that $p_i = c_i - c_{i-1}$. Refer to this spreadsheet for simulated concrete numbers for the case where $x_i = i$.

12.1.2 O(1) Convergence

Consider two consecutive steps, and condition the subsequent random walk on a, b, c, d being the four participants chosen in these steps where $a < b < c < d$.

If a, b are chosen in the first step, then the solution that remains at the end of the two steps will be c, regardless of what the solution was at the beginning of the two steps

If c, d are chosen first, then the solution that remains at the end of the two steps will be b, regardless of what the solution was at the beginning of the two steps

Removing the conditioning, all dependence from the past is lost with probability $\frac{1}{3}$ in two round. Hence, convergence to stationary distribution in $O(1)$ steps.

12.2 Distortion

We can compute distortion by computing the social choice of each solution, x_i. This can be computed as the sum of distances from chosen solution. Specifically, $\text{cost}(i) = \frac{i(i-1)}{2} + \frac{(n-i+1)(n-i)}{2}$. Expected social cost can then be define as follows:

$$\mathbb{E}[\text{cost}] = p_i(\frac{i(i-1)}{2} + \frac{(n-i+1)(n-i)}{2})$$

It can be shown that 1.208 is the max expected distortion on general median spaces. This result is particularly interesting because the protocol does not need to know that the underlying space is. Agents also do not need to know it - they only need to do Nash bargaining.

12.2.1 Worst Case Distortion for Dictatorship

Let's take $N + 1$ nodes representing the different agents' ideal points. Let's take a star graph with x_0 being the node in the middle and it's distance to every other node is 1. More formally,

$$d(x_0, x_i) = 1$$

for $x_i > 0$

$$d(x_i, x_j) = 2$$

for $x_i, x_j > 0$ and $i \neq j$

The best attainable total cost is by picking x_0 which results in a cost of $\sum_{i=1}^{N} 1 = N$. The cost of all other $x_i = 2N - 1$. This is because distance to x_0 is 1 and distance to every other node is 2. Hence, the expected cost of a random dictator is as follows

$$\text{Expected cost of random dictator} = \frac{N(2N-1)+N}{N+1}$$

Distortion can then showed to be as follows.

$$\text{Distortion} = \lim_{N\to\infty} \frac{N(2N-1)+N}{N(N+1)} \approx 2$$

A similar method on star graphs can be used for the case of sequential deliberation to show that the distortion is close to 1.

13 Markets

13.1 Fisher Market

13.1.1 Setup

We first setup a Fisher Market as follows.

Assume there are N agents M goods. Each agent i has an endowment B_i. This is basically some amount of money the agent gets to spend. Note that this money has no utility outside the market (specific to Fisher Market). For each of the M goods, C_i units of it exists.

We then say that each Agent i has a utility function $U_i(x_1, ..., x_m)$ on goods received, where x_i is the units of good i received. Additionally, we will also assume that $B_i = 1$ for N agents and $C_j = 1$ for all M goods. Note that the goods are still divisible. Also, assume that the agent utility function is linear. Specifically,

$$U_i(x_1, ..., x_m) = \sum_j U_{ij} x_j \tag{1}$$

where $U_{ij} \geq 0$

13.1.2 Market Clearance

The existence of market clearing prices entails the following. Specifically, there exist prices p_j for goods and and an allocation $x_i j$ (amount of good j given to agent i) that fulfil the following properties.

All goods are consumed $\forall j \sum_i x_{ij} = 1$

Every agent exhausts his/her endowment $\forall i \sum_i p_j x_{ij} = 1$

Every agent achieves max utility $\sum_j U_{ij} x_j = \max_{z_1, z_2, ..., z_m} \sum_j U_{ij} z_j$ s.t. $z \geq 0$ & $\sum_j p_j z_j \leq 1$

Theorem 13.1 *For the Fisher Market as defined above, a market clearing solution always exists. The allocation x_{ij} can be found by solving the following equation.*

$$\max_{x_{ij}} \sum_i \log(\sum_j U_{ij} x_{ij}) \tag{2}$$

s.t. $\forall \sum_i x_{ij} \leq 1$ *&* $\forall i \forall j x_{ij} \geq 0$

References

[1] B. Fain, A. Goel, and K. Munagala Sequential Deliberation Slides.