MS&E 235: Lecture 2
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Lecture 2: Games, Equilibria & Introduction to Auctions

This lecture focuses on Games, Nash Equilibria, Dominant Strategies and Second Price Auctions. These will all be essential tools for understanding internet advertising.

Games

We define a game to be an interaction between rational agents, or players. Imagine N players $1, 2, ..., N$. Each player i chooses an action (or move), m_i, from his/her set S_i of feasible actions. For every combination of player actions, there is a payoff, $\pi_i(m_1, m_2, ..., m_N)$, specified for each player i.

Example: Rock Paper Scissors

$N = 2$ (2 players)
$S_1 = S_2 = S_i = \{\text{Rock, Paper, Scissors}\} = \{R, P, S\}$
Profit matrices:

Player 1:

π_1	R	P	S
R	0	1	-1
P	-1	0	1
S	1	-1	0

Player 2:

π_2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

In these matrices, Player 1's actions are denoted as columns, and Player 2's actions are rows. For example, if Player 1 plays S, and Player 2 plays R, Player 1's profit is -1. Equivalently, $\pi_1(S, R) = -1$. We will come back to this game.

Nash Equilibrium

A set of actions, $(\bar{m}_1, \bar{m}_2, ..., \bar{m}_N)$ is a *Nash Equilibrium* if for all players, i, and all moves $p \in S_i$,

$$\pi_i(\bar{m}_1, \bar{m}_2, ..., p, ..., \bar{m}_N) \leq \pi_i(\bar{m}_1, \bar{m}_2, ..., \bar{m}_N)$$

In other words, a set of actions is a Nash Equilibrium if, when this set of actions is being played, no player has any incentive to unilaterally deviate from his/her current action.

Example: Prisoner's Dilemma

Two thieves are captured, and held in separate cells. Each thief is told that he has two options: remain silent, S, or turn witness, W. Their payoff matrices are as follows:

π_1	S	W
S	-1	0
W	-10	-5

π_2	S	W
S	-1	-10
W	0	-5

Where Player 1's actions are the columns and Player 2's actions are the rows. For example, if Player 1 remains silent and Player 2 turns witness, then $\pi_1 = -10$ and $\pi_2 = 0$.

The payoff matrices define the game as follows: if both thieves remain silent, they are brought up on lesser charges, and they each serve a 1 year sentence. If one thief remains silent, and the other thief turns witness, then the silent thief is convicted, and serves a 10 year sentence, while the thief that turned witness is allowed to go free. If both thieves turn witness, then they are both convicted, and each serves a 5 year sentence.

Although it seems like the thieves should both remain silent, the unfortunate Nash Equilibrium of this problem is: (W, W). Both thieves turn witness.

To see why this is a Nash Equilibrium, consider the game when both players remain silent. Player 1 can improve his payoff by turning witness. If Player 1 turns witness, and Player 2 remains silent, then Player 2 can improve his payoff by turning witness. If both players are witnesses, then neither player has any incentive to change his action, and thus (W, W) is a Nash Equilibrium.

Example: Cars Waiting at Railroad Tracks

Consider the scenario of cars stopped at a railroad crossing. While the train is going by, more cars pull up. As a driver, you have the choice of either: getting in line behind the existing cars, or going into the left lane (the oncoming traffic lane) and merging (a.k.a. cutting) back into the right lane after the train goes by. As a rational player, you would choose to go into the left lane.

This is a Nash Equilibrium, but like the prisoner's dilemma, leads to a clearly suboptimal outcome.

Example: Rock Paper Scissors

In Rock Paper Scissors, there is no pure strategy Nash Equilibrium. If $m_1 = P$ and $m_2 = P$ then Player 1 can improve his payoff by playing S. But if Player 1 plays S, Player 2 would rather play R. If Player 2 plays R then Player 1 would rather play P, and so on.

For games such as this, there are *Mixed Strategy Nash Equilibria.*

Mixed Strategy Nash Equilibrium

A player who plays more than one action with positive probability is said to play a *mixed action.* The same definition for Nash Equilibrium applies when mixed actions are played. In this case it is called a *Mixed Strategy Nash Equilibrium.* A Nash Equilibrium where players play only pure strategies (what was previously called "Nash Equilibrium"), may also be referred to as *Pure Strategy Nash Equilibrium*, and is actually a subset of Mixed Strategy Nash Equilibrium.

Example: Rock Paper Scissors

Lets say Player 1 plays R with probability 0.5 and S with probability 0.5 (and P with probability 0). This is a mixed action and is denoted $m_1 = (0.5, 0.5, 0)$.

Q: Is $m_1 = (0.5, 0.5, 0), m_2 = (0.5, 0.5, 0)$ a Mixed Strategy Nash Equilibrium?

A: No, because Player 1 can improve his/her payoff by playing $m_1 = (0, 1, 0)$.

In fact, Rock Paper Scissors has a unique Nash Equilibrium: $m_1 = m_2 = (1/3, 1/3, 1/3)$.

Example: Cooperative Matching Pennies

$N = 2$
$S_1 = \{Heads, Tails\} = \{H, T\}$
$S_2 = \{H, T\}$

Payoff Matrix for both players:

$\pi_{1,2}$	H	T
H	1	0
T	0	1

In this game, each player has a penny, and they secretly choose heads or tails. They both reveal their choices at the same time, and if they have chosen the same side of the coin, then they both win. If the coins do not match, then they both lose.

Q: Find all the Nash Equilibria of this game.

A: (H,H), (T,T), and $((0.5,0.5),(0.5,0.5))$

Dominant Strategy Equilibrium

A set of actions, $(\bar{m}_1, \bar{m}_2, ..., \bar{m}_N)$, is a *Dominant Strategy Equilibrium* if for all players i and all actions $\gamma_1, \gamma_2, ..., \gamma_N$ for the N players,

$$\pi_i(\gamma_1, \gamma_2, ..., \gamma_N) \leq \pi_i(\gamma_1, \gamma_2, ...\gamma_{i-1}, \bar{m}_i, \gamma_{i+1}, ..., \gamma_N)$$

In other words, a player's best move does not depend on what the other players do in a Dominant Strategy Equilibrium.

Example: Prisoner's Dilemma

The Prisoner's Dilemma is an example of a game that has a Dominant Strategy Equilibrium. No Matter what Player 2 plays, Player 1 is always better off turning witness, and likewise for Player 2.

Auctions

An auction is a game where players bid on items, winners are chosen, and then the players pay some amount to the auctioneer. Auctions are very important for internet commerce because this is how keyword ad spots are sold. As we saw in lecture 1, you cannot simply choose a price for a keyword and ask for that price. But the notion of an auction is quite vague, and in order to implement an auction, a mechanism must be chosen. Here are some typical kinds of auctions.

First Price Auction

A First Price Auction is a type of sealed-bid auction, which means that players bid only once, and do so without knowledge of other players' bids. In a First Price Auction, players submit bids, b_i, and the player with the highest bid wins the item. The winning player then pays the amount that he bid to the auctioneer, and the other players pay nothing. Players, as usual, attempt to maximize their payoffs. Therefore, the player with the highest bid can increase his payoff by bidding just above the second highest bidder. That way, he still wins the item, but pays as little as possible.

Example from class: Students place bids on Professor Goel's phone. Student 1 bids \$60, Student 2 bids \$50 and Student 3 bids \$10. If it is a First Price Auction, then Professor Goel should award the phone to Student 1 for a price of \$60. But Student 1 could have improved his payoff by bidding lower. For instance, if he had bid \$51, he still would have won, but would have paid a lower price for the same phone.

As the example illustrates, the outcome is not dominant and introduces instabilities into the market because players do not know what other players' bids are, and thus must speculate. For this reason, First Price Auctions are generally not advisable in large scale systems. A more stable type of auction is the *Second Price Auction*.

Second Price Auction

A Second Price Auction (also known as a Vickrey Auction) is another type of sealed-bid auction. In a Second Price Auction, players submit bids, b_i, and the highest bidder wins the item. This time the winner pays the amount of the *second* highest bid to the auctioneer. A desirable property of Second Price Auctions, is that there is a dominant strategy equilibrium. The dominant strategy equilibrium is when every player bids his/her true valuation.

Consider the example from class again, where students bid on Professor Goel's phone. If this is a second price auction, then the phone is awarded to Student 1 (the highest bidder) at a price of \$50 (the next highest bid). Unlike the First Price Auction, Student 1 has no incentive to deviate from his bid. If he had bid lower than \$50, he would have lost, and thus would be worse off. If he had bid anywhere above \$50, he would still have won, and paid the same amount. Likewise, Student 2 and 3 are at least as well off as they would have been by bidding any other amounts. Student 2 could have outbid Student 1, and won the phone, but he would have had to pay \$60. If he is only willing to pay \$50, then this will not make him better off.

eBay is essentially a second price auction. Users place bids, and when the auction is over, eBay awards the item to the highest bidder at a price that is marginally higher than the next-highest bid.

Another example is the English Auction, the type of auction traditionally used by auction houses. Shoppers bid up the item incrementally. When the second highest bidder no longer wishes to bid, the only one left is the highest bidder, who wins the item and pays a small amount more than the second highest bid. Notice that this is not a first price auction because the bidders do not submit their bids simultaneously, but are given the option to outbid after each bid is placed.

Next time: Ad Spot Auctions