Finance Problem Statement

1 Super-Replication of a Target Payoff with Options and Bonds

A portfolio manager aims to construct a portfolio to **super-replicate** the payoff of a target financial instrument, given different market states. The available instruments are European call options, put options, and a risk-free bond.

The target payoff $\mathbf{y} \in \mathbb{R}^n$ represents the desired payoff across n market states, where each entry y_i is the target payoff in state i. The available instruments have known payoffs across these states:

- A **call option** on an underlying stock with strike price K_c.
- A **put option** on the same stock with strike price K_p.
- A **risk-free bond** that pays 1 unit in all states.

Define the payoff matrix $\mathbf{A} \in \mathbb{R}^{n \times 3}$, where each column corresponds to the payoffs of the call option, put option, and bond across the n states.

The portfolio manager wants to determine the optimal holdings $\mathbf{x} = [x_{\text{call}}, x_{\text{put}}, x_{\text{bond}}]^\top$ in each instrument to achieve a super-replication, meaning that the portfolio's payoff should **meet or exceed** the target payoff in each state.

The costs for each unit of the instruments are given by $\mathbf{c} = [c_{\text{call}}, c_{\text{put}}, c_{\text{bond}}]^\top$, with the objective to minimize the total cost of the portfolio.

Formulate the linear program that will determine the optimal values for $x_{\text{call}}, x_{\text{put}}, x_{\text{bond}}$ to super-replicate the target payoff while minimizing costs.

Constraints

1. **Super-replication constraint**: In each state i, the portfolio payoff should meet or exceed the target payoff:

$$A_{i,\text{call}} \cdot x_{\text{call}} + A_{i,\text{put}} \cdot x_{\text{put}} + A_{i,\text{bond}} \cdot x_{\text{bond}} \geq y_i \quad \text{for all } i = 1, \ldots, n.$$

2. **Non-negativity constraint**: All investments must be non-negative, i.e., $x_{\text{call}}, x_{\text{put}}, x_{\text{bond}} \geq 0$.

Objective: Minimize the total cost of the portfolio, given by $\mathbf{c}^\top \mathbf{x}$.

Numerical Example

Consider a scenario with three states for the underlying stock price:

- **States**: Low, Medium, High.
- **Target payoff**: $\mathbf{y} = [5, 15, 25]^\top$ (representing a payoff that increases as the stock price rises).

Assume the payoffs of each instrument across the states are as follows:

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} \text{Call Payoff} & \text{Put Payoff} & \text{Bond Payoff} \end{array} \\ \begin{bmatrix} \phantom{\text{Call Pa}}0\phantom{\text{ayoff}} & \phantom{\text{Put Pa}}5\phantom{\text{ayoff}} & \phantom{\text{Bond Pa}}1\phantom{\text{ayoff}} \\ 5 & 0 & 1 \\ 10 & 0 & 1 \end{bmatrix} \end{array}$$

- **Costs**: $\mathbf{c} = [3, 2, 1]^\top$ (cost per unit of call option, put option, and bond, respectively).

Using this data, set up and solve the linear program.

2 Arbitrage Detection

A trader is analyzing a market with three assets, each with different prices and returns across three possible future states of the world. The trader wants to identify whether an arbitrage opportunity exists—specifically, a combination of asset holdings that guarantees a non-negative payoff in every state and a strictly positive payoff in at least one state, while starting with zero or less initial investment.

The assets have known returns across the states, given by the following payoff matrix $\mathbf{R} \in \mathbb{R}^{3\times3}$, where R_{ij} is the payoff of asset j in state i. x_j represent the amount invested in asset j,

Suppose the market data for three assets is as follows:

$$\mathbf{R} = \begin{bmatrix} 1.1 & 0.9 & 1.2 \\ 1.0 & 1.05 & 0.95 \\ 1.2 & 1.0 & 1.3 \end{bmatrix}$$

Problem

1. **Formulate a linear program** to determine whether there exists an arbitrage opportunity. Specifically, find a vector $\mathbf{x} = [x_1, x_2, x_3]^\top$ representing amounts invested in the assets that satisfy the following conditions:

   - **Non-negative payoff in each state**: In each state i, the portfolio's payoff should be non-negative:

     $$R_{i1} \cdot x_1 + R_{i2} \cdot x_2 + R_{i3} \cdot x_3 \geq 0.$$

   - **Negative initial investment**: The initial investment should be -1:

     $$x_1 + x_2 + x_3 = -1.$$

2. **Interpret the results**: If a feasible solution $\mathbf{x}$ exists, it indicates an arbitrage opportunity. Explain why this is the case. If no solution exists, explain what this implies about the market.