MS&E 114/214 Autumn 2024 Homework 2

Submission is due **12:00 pm Oct 20.** When submitting, please attach the relevant Python scripts and/or Excel files to enable us to give partial credits, and share the link to your Colab and/or Excel files.

Problem 1

Consider the following scenario: Four roommates rent a house with four rooms. Their valuations for each room are given to you; each person's valuations sum up to the full rent for the house. For each of the three valuation matrices in the Excel spreadsheet provided to you, compute the matching that maximizes valuations by solving a linear program in Excel. For each of the three valuation matrices, generate some set of envy-free prices for each setting, assuming the matching is the one that you found. These prices must be generated using values from the sensitivity report for each case, and you need to clearly show how you derived the prices from the sensitivity reports. The envy-free prices don't need to maximize the minimum utility, but they do need to sum to the total rent. Note that the dual values in the sensitivity report may not sum to the total rent and, hence, may need to be adjusted up or down. For any one valuation matrix of your choosing, write down the complementary slackness conditions and explain how they are being satisfied (i.e., is the constraint tight, the variable 0, or both).

Problem 2

Use the same data as the previous problem.

(a) For the third valuation matrix, compute the envy-free prices that maximize the minimum utility.

(b) Suppose the first two roommates are twins, and are not going to be envious of each other's assignments. How would you change your LP formulation to take this into account (do not solve; just let us know what the change would be).

(c) True or false: For any instance of the fair rent division problem, the envy-free prices that maximize the minimum utility must give the same utility to each roommate. Provide a formal proof.

(d) True or false: For any instance of the fair rent division problem, there exist envy-free prices that are non-negative. Provide a formal proof.

(e) *(EC for 114, regular problem for 214)* Show that for any instance of the fair rent division problem, there exists a matching with total valuation at least as much as the total rent for the house. *[There is a hint on the slides.]*

(f) *(EC for 114 and also for 214)* The duality-based proof in class for the existence of envy-free prices is incomplete since it does not find prices that sum up to the total rent. Complete this proof by showing that the prices can be adjusted up or down so that they sum up to the total rent without violating any other constraints in the problem, and without giving any player negative utility.

Problem 3

Consider a knapsack problem where a thief has a knapsack with a maximum volume U and weight limit W. The thief is in a warehouse with N goods. Each good i has a weight w_i, a volume u_i, and a value v_i.

Additionally, we assume each good to be infinitely divisible, implying not only that we can take fractions of goods, but that we can pack the goods into our knapsack such that the entire space is filled. The thief wants to maximize the value in the knapsack subject to the two capacity constraints.

Solve the following instance of the LP with four items in Excel. The maximum volume is 100, and the maximum weight is 30. The following table shows the volume, weight and value of the items.

	Volume	Weight	Value
Item 1	60	40	3
Item 2	40	10	5
Item 3	20	20	1
Item 4	20	50	8

Problem 4

Consider the following LP:

$$
\begin{gathered}
\text{Maximize} \quad 3x_1 + 2x_2 + x_3 \\
\text{s.t.} \quad x_1 - x_2 + x_3 \leq 4 \\
2x_1 + x_2 + 3x_3 \leq 6 \\
-x_1 + 2x_3 \leq 3 \\
x_1 + x_2 + x_3 \leq 8 \\
x \geq 0
\end{gathered}
$$

(a) State the dual problem.

(b) Suppose that you suspect that the vector $(x_1, x_2, x_3) = (0, 6, 0)$ is optimal for the primal problem. Use the complementary slackness conditions to solve the dual problem, and then show that your suspicion is correct.

Problem 5

(a) Consider the Python notebook for participatory budgeting that we provide. It uses only the first 500 voters, for efficiency (by using *preferred_budget_all_voters[0:500]*). For each of the two methods provided, plot the time for solving the problem as a function of the number of voters N, where N takes values from $\{100, 300, 500, 1000, 1500, 2000\}$. Does the time appear to be scaling linearly?

(b) The city of Austin now tells you that the amount allocated to the first project must be at least 1050. Find the aggregate budget that maximizes the total overlap utility for the first 500 voters taking this additional constraint into account.