MS&E 214
Optimization via Case Studies

Week 6: Finance

Ashish Goel

(based on slides by Professor Benjamin Van Roy)

Roadmap

- Finished two case studies
  - Social Choice and Welfare
    - Matchings
    - Participatory Budgeting
    - Scoring Rules
    - Fair Division
    - Walrasian Equilibrium
  - Machine Learning
    - Regression
    - Pattern Classification
- Techniques
  - LPs (Everywhere)
  - Quadratic Programming (Regression, SVMs)
  - Basic Feasible Solutions (Matchings)
  - Duality (Fair Division)
  - Max-min and Min-max objective functions (Fair Division, Regression, Pattern Classification, Participatory Budgeting)
  - Still to come: The logarithmic trick, Convex Optimization in general, Network Flows, Revisit Duality, Revisit Basic Feasible Solutions
- Next: Finance
  - Options (as a linear payoff function)
  - Designing portfolios to cover a specific liability under diverse market scenarios (LP)
  - Arbitrage (LP); Currency arbitrage (network flows with the logarithmic trick)
  - Portfolio optimization (Quadratic Programming)

Two different approaches to finance in this class

- Assume that you can enumerate all future market scenarios (useful for financial derivatives whose values depend on a small number of well defined market variables), and enumerate your future payoffs and liabilities in all possible scenarios

vs

- Assume that the market is complex, and different products interact in ways that are impossible (or too hard) to fully characterize. Hence, we use past statistical measures (such as average returns and variance of returns of different stocks, the correlations between different stocks) to optimize a risk-adjusted measure of expected future returns

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Today

- Assume that you can enumerate all future market scenarios and enumerate your future payoffs and liabilities in all possible scenarios. Useful for financial derivatives whose values depend on a small number of well defined market variables.

Next lecture

vs

- Assume that the market is complex, and different products interact in ways that are impossible (or too hard) to fully characterize. Hence, we use past statistical measures (such as average returns and variance of returns of different stocks, the correlations between different stocks) to optimize a risk-adjusted measure of expected future returns

Contingent Claims

- A context and application area for Linear Programs

- A contingent claim is a contract
  - Receive a **payoff** in the future depending on an uncertain outcome, e.g. the price of a stock in the future
  - May pay a **price** to purchase the contract
  - Current money is not the same as future money

- Examples
  - Insurance
  - Negotiated contract with contingencies
  - Stocks, bonds, options, and other derivatives

- Mathematical representation
  - Enumerate possible outcomes 1, 2, ..., M
  - Specify outcome-contingent payoffs $\mathbf{a} \in \Re^M$

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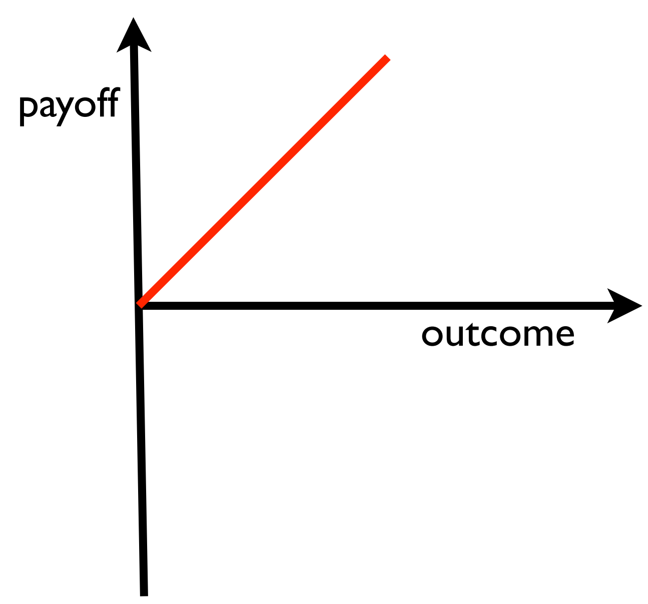
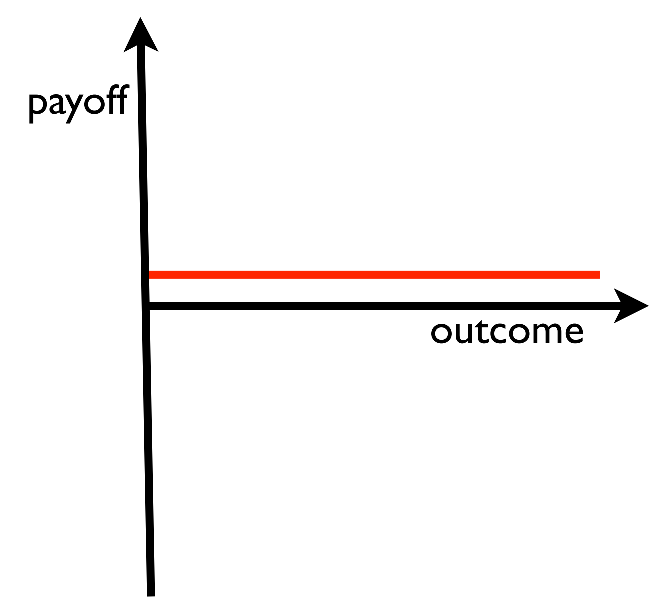
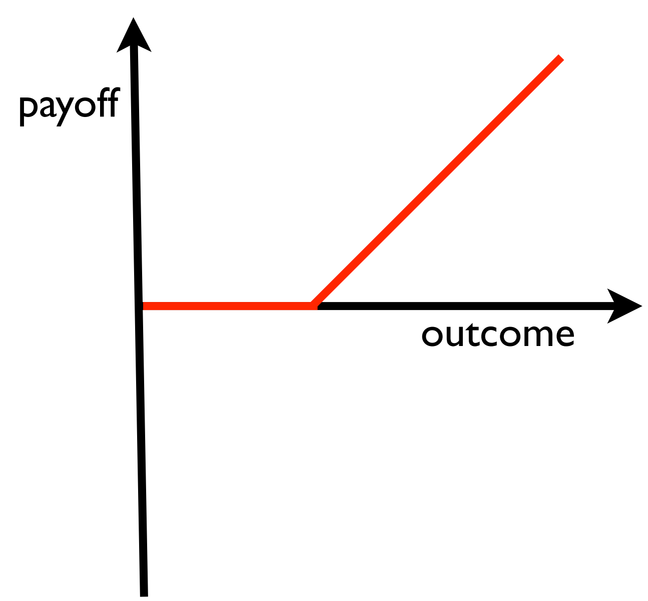
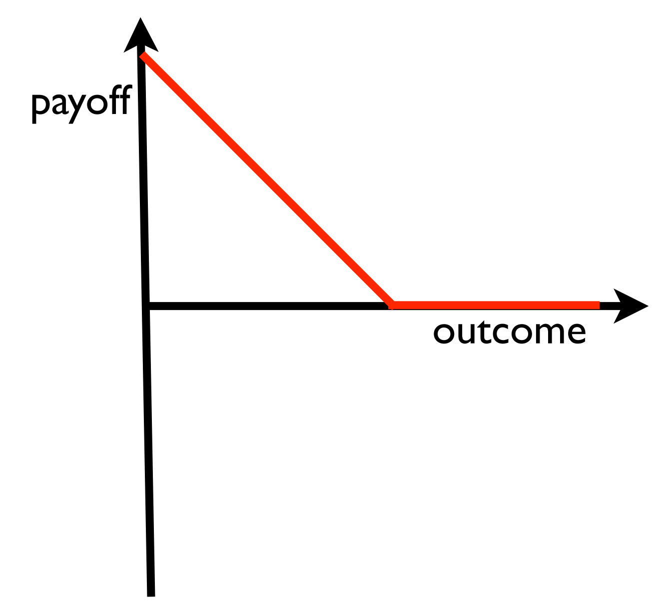
Stocks and Bonds

- Assume one year holding period

	stock	zero-coupon bond
(Current) price	p_1	p_2
outcomes	future stock price = 1, ..., M	future stock price = 1, ..., M
(Future) payoff vector	$\mathbf{a}^1 \in \Re^M$	$\mathbf{a}^2 \in \Re^M$
illustration		

European Calls and Puts

	European call option	European put option
price	p_3	p_4
expiration date	1 year	1 year
strike price	\$40	\$60
outcomes	future stock price = 1, ..., M	future stock price = 1, ..., M
payoff vector	$\mathbf{a}^3 \in \Re^M$	$\mathbf{a}^4 \in \Re^M$
illustration		

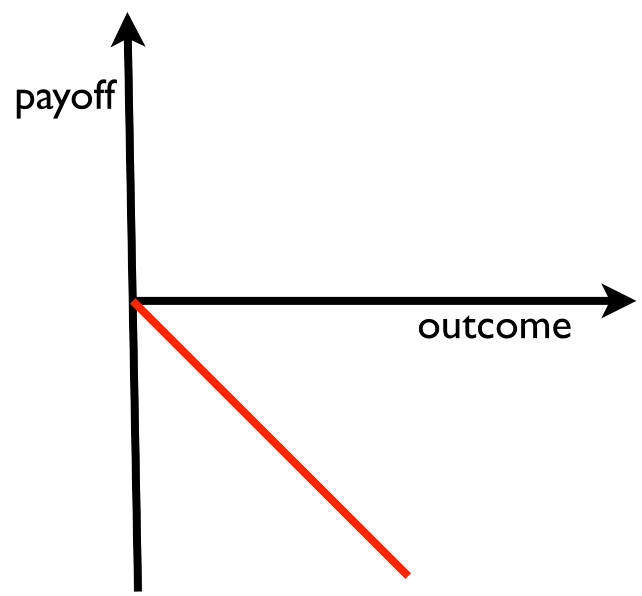
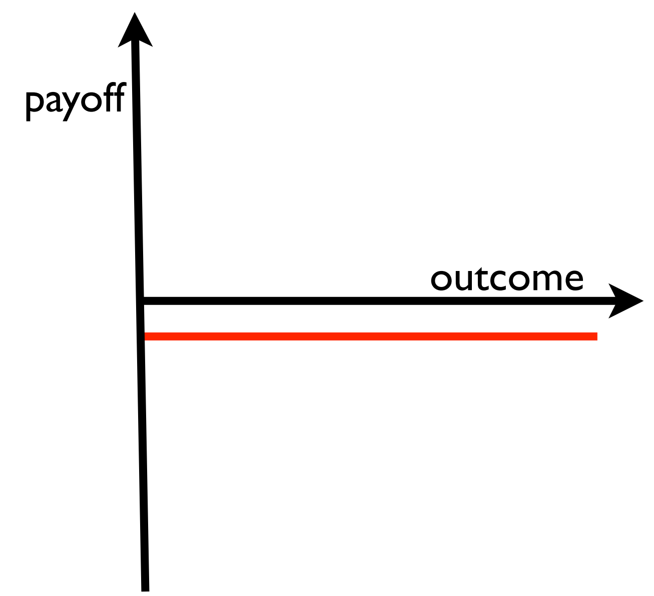
Call and Put Options

t = Maturity date
s = Strike Price
z = future price of stock on maturity date

- Call option: Payoff = $\max\{0, z-s\}$
- Put option: Payoff = $\max\{0, s-z\}$

Short Selling

- Broker borrows/sells contingent claim you don't have

	short sell stock	short sell zero-coupon bond
price	$-p_1$	$-p_2$
outcomes	future stock price = 1, ..., M	future stock price = 1, ..., M
payoff vector	$-\boldsymbol{a}^1$	$-\boldsymbol{a}^2$
illustration		

- Modeling simplifications: no transaction costs or margin requirements

Call and Put Options: Complex Example

A broker buys a single put option on a stock in corporation XYZ, expiring at time t. She also short-sells a call option on the same stock, expiring at the same time t. The strike price on both options is \$50. She also buys one unit of this stock which she will liquidate at time t. If the future price of the stock is z, what is her payoff at time t?

0

\$50 - z

z - \$50

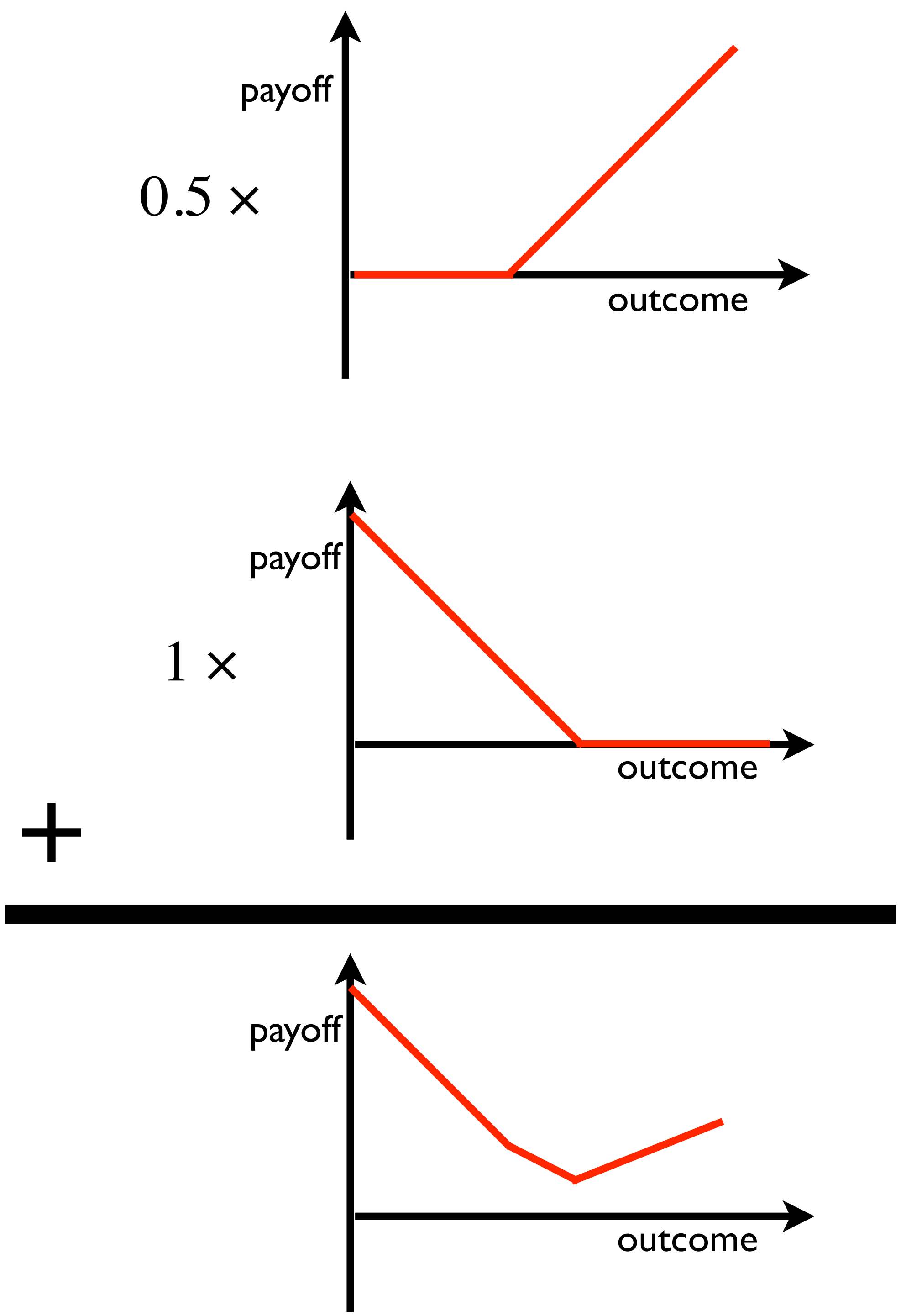
\$50

z

Markets and Portfolios

- N contingent claims, M payoff-relevant outcomes
- Payoff matrix (future) $\mathbf{P} \in \Re^{M \times N}$
- Market prices (current) $\rho \in \Re^{N}$ (row vector)
- Portfolio vector $\mathbf{x} \in \Re^{N}$
- Portfolio payoff $\mathbf{Px} \in \Re^{M}$
- Portfolio price $\rho \mathbf{x}$

Example



$$P = \begin{bmatrix} 0 & 8 \\ 0 & 7 \\ 0 & 6 \\ 0 & 5 \\ 0 & 4 \\ 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & 0 \\ 7 & 0 \\ 8 & 0 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 3 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$Px = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2.5 \\ 2 \\ 1.5 \\ 2 \\ 2.5 \\ 3 \\ 3.5 \\ 4 \end{bmatrix}$$

$$\rho x = 3.5$$

Replication

- Liabilities $\mathbf{b} \in \Re^M$
- Replicating Portfolio $\mathbf{P}\mathbf{x} = \mathbf{b}$
- Price of Replication $\rho\mathbf{x}$

Example of Replication

A banker has sold a call option on a stock with strike price $50, maturing in one year. The banker has access to a zero-coupon bond with payoff $1 cost 0.95 maturing in one year. The banker also can buy the stock at the current price $40 and keep it. And the banker can also buy a put option on the same stock with a strike price of $50 at cost $1. Design a portfolio that replicates the payoff from the call option that the banker has sold.

(Of minimum current cost)

Solve using excel.

Digression: Put-Call Parity and Intro to Arbitrage

If strike price is s, and price of zero-coupon bond is b, price of stock right now is z, price of call option is c and price of put option is p, then

Payoff of put option + payoff of stock - payoff of z units of bond = payoff of call option.

Hence,

p + z - bs = c (know as put-call parity)

Violation will lead to arbitrage (ability to make positive money now with a non-positive liability in the future for all possible scenarios)

Violated in the previous scenario (put-call parity implies p >= bs-z)

=> hence arbitrage

Solve excel again with cost of put option = $8. Resulting cost must be the price of the call option for there to be no arbitrage

Super-Replication

- What if there is no replicating portfolio?
  - Incomplete market

- Super-Replication $\mathbf{P}\mathbf{x} \geq \mathbf{b}$

- Minimize price

$$\begin{array}{lrcl} \text{minimize} & \rho\mathbf{x} & & \\ \text{subject to:} & \mathbf{P}\mathbf{x} & \geq & \mathbf{b} \end{array}$$

Example of Super-Replication

A banker has sold a call option on a stock with strike price \$50, maturing in one year. The banker has access to a zero-coupon bond with cost 0.95 maturing in one year. The banker also can buy the stock at the current price \$40 and keep it. And the banker can also buy a put option on the same stock with a strike price of \$45 at cost \$1. Design a portfolio that replicates the payoff from the call option that the banker has sold.

Solve using excel.

Example of Super-Replication

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Super-Replicates

Solve using excel.

Example of Super-Replication

A banker has sold a call option on a stock with strike price \$50, maturing in one year. The banker has access to a zero-coupon bond with cost 0.95 maturing in one year. The banker also can buy the stock at the current price \$40 and keep it. And the banker can also buy a put option on the same stock with a strike price of \$45 at cost \$1. Design a portfolio that super-replicates the payoff from the call option that the banker has sold. (Px >= b as opposed to Px = b)

Solve using excel.

Which scenarios to use: In this case, since the payoffs are all piecewise linear functions, it is enough to ensure that every "change point" is represented and at least one scenario smaller than all the change points and at least one scenario larger than all the change points

Arbitrage

- **Def.** arbitrage opportunity

$$\mathbf{x} \in \Re^N \quad \text{s.t.} \quad \rho\mathbf{x} < 0 \quad \textit{and} \quad \mathbf{Px} \geq 0$$

- Most lucrative arbitrage opportunity

$$\begin{array}{lll} \min & \rho\mathbf{x} & \\ \text{s.t.} & \mathbf{Px} \geq & 0 \end{array}$$

- Is there a problem with this?

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- Is there a problem with this?

It is always infeasible

It always has an unbounded optimum

Both of the above

It does not identify an arbitrage opportunity

Arbitrage

- **Def.** arbitrage opportunity

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$$\begin{array}{lll} \min & \rho\mathbf{x} & \\ \text{s.t.} & \mathbf{P}\mathbf{x} \geq & 0 \end{array}$$

$$\mathbf{x} \to \infty$$

- Arbitrage opportunity that makes $1

$$\begin{array}{llll} \min & \ldots & & \\ \text{s.t.} & \rho\mathbf{x} & = & -1 \\ & \mathbf{P}\mathbf{x} & \geq & 0 \end{array}$$

Minimizing Shares Traded

$$\begin{array}{llcl} \min & \sum_{i=1}^{N} |x_i| & & \\ \text{s.t.} & \rho \mathbf{x} & = & -1 \\ & \mathbf{P}\mathbf{x} & > & 0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{llcl} \min & \sum_{i=1}^{N} (x_i^+ + x_i^-) & & \\ \text{s.t.} & \rho(\mathbf{x}^+ - \mathbf{x}^-) & = & -1 \\ & \mathbf{P}(\mathbf{x}^+ - \mathbf{x}^-) & \geq & 0 \\ & \mathbf{x}^+, \mathbf{x}^- & \geq & 0 \end{array}$$

Structured Products

- US manufacturer with offer from retailer in China
  - 100,000 units of telematics system
  - Delivery in three months
- Price of Yuan three months from now influences
  - Shipping and assembly decisions
  - Resulting profit
- Risky project: may or may not be profitable
- Purchase **structured product** that covers **liabilities**
  - Can be **replicated** by trading bonds, Yuan, and options
- Guaranteed positive profit, received immediately