MS&E 114/214 Session 2

LP Duality

n variables, m constraints

Maximize $c^T x$

s.t. $Ax \leq b$

$x \geq 0$

Primal standard form

m variables, n constraints

Minimize $b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

Dual standard form

PRIMAL

DUAL

Complementary Slackness: Definition

- Consider optimum primal/dual solutions
- For every constraint in the primal, either the constraint is binding or the corresponding dual variable is 0
  - Concisely written as $(Ax - b)y = 0$
- For every constraint in the dual, either the constraint is binding or the corresponding primal variable is 0
  - Concisely written as $(A^T y - c)x = 0$

[Binding constraint: satisfied **exactly**]

LP Duality example

Maximize $2x_1 + x_2$

s.t.

$x_1 - x_2 \leq 5$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Minimize $5y_1 + 4y_2$

s.t.

$y_1 + y_2 \geq 2$

$-y_1 + 2y_2 \geq 1$

$y_1, y_2 \geq 0$

PRIMAL

Optimum solution: ($x_1 = 4$, $x_2 = 0$)
Objective: 8
Constraints binding: Only second

DUAL

Optimum solution: ($y_1 = 0$, $y_2 = 2$)
Objective: 8
Constraints binding: Only first

Consider the following linear program:

$$\begin{aligned} \textbf{max} \quad & 3s + 2t + 4u \\ \textbf{\textit{s.t.}} \quad & s + t - u \leq 20 \\ & s - t + 2u \leq 4 \\ & -s + 2t + u \leq 3 \\ & s, t, u \geq 0 \end{aligned}$$

(a) Rewrite the program in the following form by specifying c, x, A and b:

$$\begin{aligned} \textbf{max} \quad & c^T x \\ \textbf{\textit{s.t.}} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

(a) With $c = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$, $x = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$, $b = \begin{bmatrix} 20 \\ 4 \\ 3 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, we can rewrite the program as:

$$\begin{aligned} \textbf{max} \quad & c^T x \\ \textbf{\textit{s.t.}} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \textbf{max} \quad & 3s + 2t + 4u \\ \textbf{s.t.} \quad & s + t - u \leq 20 \\ & s - t + 2u \leq 4 \\ & -s + 2t + u \leq 3 \\ & s, t, u \geq 0 \end{aligned}$$

(b) The optimal solution of this program is $x = \begin{bmatrix} 11 \\ 7 \\ 0 \end{bmatrix}$.

(i) What is the optimal value of this LP?

(ii) Determine which constraints are binding, and which are not.

(b) (i) The optimal value is $3 \times 11 + 2 \times 7 + 4 \times 0 = 47$

(ii) Let's see which constraint are binding at the optimal solution:

- $1 \times 11 + 1 \times 7 - 1 \times 0 = 18 < 20$: constraint 1 is non-binding
- $1 \times 11 - 1 \times 7 + 2 \times 0 = 4$: constraint 2 is binding
- $-1 \times 11 + 2 \times 7 + 1 \times 0 = 3$: constraint 3 is binding

$$\begin{aligned} \textbf{max} \quad & 3s + 2t + 4u \\ \textbf{\textit{s.t.}} \quad & s + t - u \le 20 \\ & s - t + 2u \le 4 \\ & -s + 2t + u \le 3 \\ & s, t, u \ge 0 \end{aligned}$$

(c) Fill in the blanks: the associated dual program is:

$$\begin{aligned} \textbf{min} \quad & \_\_\_\ y_1 + \_\_\_\ y_2 + 3y_3 \\ \textbf{\textit{s.t.}} \quad & y_1 + \ \_\_\_\ y_2 - y_3 \ge 3 \\ & y_1 + \_\_\_\ y_2 + 2y_3 \ge 2 \\ & \_\_\_\ y_1 + 2y_2 + \_\_\_\ y_3 \ge 4 \\ & y_1, y_2, y_3 \ \_\_\_\ 0 \end{aligned}$$

LP Duality

PRIMAL	DUAL
n variables, m constraints	m variables, n constraints
Maximize $c^T x$	Minimize $b^T y$
s.t. $Ax \le b$	s.t. $A^T y \ge c$
$x \ge 0$	$y \ge 0$
Primal standard form	**Dual standard form**

PRIMAL **DUAL**

(c) The associated dual program is:

$$\begin{aligned} \textbf{min} \quad & 20y_1 + 4y_2 + 3y_3 \\ \boldsymbol{s.t.} \quad & y_1 + y_2 - y_3 \geq 3 \\ & y_1 - y_2 + 2y_3 \geq 2 \\ & -y_1 + 2y_2 + y_3 \geq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \textbf{max} \quad & 3s + 2t + 4u \\ \textbf{\textit{s.t.}} \quad & s + t - u \leq 20 \\ & s - t + 2u \leq 4 \\ & -s + 2t + u \leq 3 \\ & s, t, u \geq 0 \end{aligned}$$

- $1 \times 11 + 1 \times 7 - 1 \times 0 = 18 < 20$: constraint 1 is non-binding
- $1 \times 11 - 1 \times 7 + 2 \times 0 = 4$: constraint 2 is binding
- $-1 \times 11 + 2 \times 7 + 1 \times 0 = 3$: constraint 3 is binding

(d) Using *only* (b)ii. and (c), what can you say about y_1, y_2 and y_3?

(d) Using (b)ii.,(c) and the complementary slackness, we know that $y_1 = 0$. We cannot conclude anything about y_2 and y_3 using only (b)ii.

$$\begin{aligned}
\textbf{min} \quad & 20y_1 + 4y_2 + 3y_3 \\
\textbf{s.t.} \quad & y_1 + y_2 - y_3 \geq 3 \\
& y_1 - y_2 + 2y_3 \geq 2 \\
& -y_1 + 2y_2 + y_3 \geq 4 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}$$

(e) Knowing that the first constraint is binding and using your previous answers, find the value of each coefficient of $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

(e) By Strong Duality, we know that the optimal value of the dual program is 47. Therefore, using the fact that the first constraint is binding and that $y_1 = 0$, we have:

$$\begin{cases} 4y_2 + 3y_3 = 47 \\ y_2 - y_3 = 3 \end{cases}$$

And we get: $y_2 = \frac{56}{7} = 8$ and $y_3 = \frac{35}{7} = 5$