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An extension and simple proof of a constrained lattice fixed point theorem

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Abstract. We extend a result of Roth dealing with fixed points of lattice mappings which satisfy certain constraints.

In this note we generalize a theorem of Roth [2], which has been applied in the study of both cooperative and combinatorial games [3, 4]. The original theorem and the extensions presented here follow in a simple way from a famous theorem due to Tarski.

Let L be a complete lattice. We will be concerned with functions f such that $f: L \to L$, and we will denote the set of fixed points of f by $L_f \equiv \{x \in L \mid x = f(x)\}$. A function f will be called *isotone* if $a, b \in L$ and $a \leq b$ implies $f(a) \leq f(b)$, and *antitone* if $a < b$ implies $f(b) \leq f(a)$. A function f is a *join antimorphism* if for any $A \subset L$, $f(\bigvee A) = \bigwedge f(A)$, where $f(A) \equiv \{f(a) \mid a \in A\}$. Note that every join antimorphism is antitone. One way in which join antimorphisms arise is the following. Let X be an arbitrary set, on which is defined an arbitrary binary relation A. For any x in X, define $D(x) = \{y \in X \mid (x, y) \in A\}$, and for any subset S of X, let $D(S) = \bigcup_{x \in S} D(x)$. Then the function $U(S) = X - D(S)$ is a join antimorphism on the lattice L of subsets of X, ordered by set inclusion. Note that if the pair (X, A) is interpreted as an abstract graph with nodes X and arcs A, then $U(S)$ is defined to be the set such that no arc connects any node in the set S with any node in the set $U(S)$. The following result is proved in [2].

THEOREM 1. *If f is a join antimorphism, then there exists an element $x \in L$ such that $x = f(f(x))$ and $x \leq f(x)$.*

This theorem and some extensions can be obtained using the following result due to Tarski [6].

THEOREM 2. *If f is isotone, then L_f is a non-empty and complete lattice relative to the same partial order.*

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This permits the following observations.

THEOREM 3. *Let h and g be functions such that h is isotone and g takes* L_h *into* L_h. *Then there exists an element* $x \in L$ *such that* $x = h(x)$ *and* $x \leq g(x)$.

Proof. Let $x = \bigwedge L_h$. Then Tarski's theorem implies $x \in L_h$ (i.e., $x = h((x))$, so $g(x) \in L_h$, and $x \leq g(x)$.

Theorem 1 follows from Theorem 3 by taking $h = f \circ f$ and $g = f$. A similar argument gives the following result.

THEOREM 4. *Let* u_1, u_2 *be functions such that* $f = u_1 \circ u_2$ *and* $g = u_2 \circ u_1$ *are both isotone. Then there exist elements* $x, y \in L$ *such that* $x = f(x) \leq u_1(y)$ *and* $y = g(y) \leq u_2(x)$.

Proof. Let $x = \bigwedge L_f = f(x)$, and $y = \bigwedge L_g = g(y)$. Then $y = u_2(u_1(y))$, so $u_1(y) = f(u_1(y))$ (i.e., $u_1(y) \in L_f$), so $x \leq u_1(y)$. Similarly, $y \leq u_2(x)$.

One way in which fixed points of the kind considered in Theorems 1 and 4 arise is in the consideration of two player non-cooperative games whose positions and moves can be described as the nodes and arcs of a graph. In a symmetric game (i.e., one in which a legal move for one player is also legal for the other) the set of winning positions is a fixed point of the kind described in Theorem 1 (cf. [4]). In an asymmetric game, (cf. Conway [1]), the sets of winning positions for each player are fixed points of the kind described in Theorem 4 (cf. [5]).

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