REPRINTED FROM

NAVAL RESEARCH LOGISTICS QUARTERLY

JUNE 1978
VOL. 25, NO. 2

OFFICE OF NAVAL RESEARCH

NAVSO P-1278

A NOTE CONCERNING ASYMMETRIC GAMES ON GRAPHS

Alvin E. Roth

University of Illinois
Urbana, Illinois

ABSTRACT

We consider a class of asymmetric two-person games played on graphs, and characterize all the positions in the game.

In this note we consider a class of asymmetric two-person games in which the players alternately choose from a set of permissible moves, and the object of the game is to make the last move. We will characterize the positions in the game as winning, losing, or drawing from the point of view of each of the players.

Formally, let (V, A) be a directed graph, where V is the (finite) set of vertices, and $A \subset VxV$ is a set of directed arcs such that $A = A_1 \cup A_2$. The players take turns choosing vertices of the graph according to the rule* that if vertex v has just been chosen by player j, then player i $(i \neq j)$ may choose any vertex x such that $(v, x) \in A_i$. That is, player i may move along arcs in A_i. Player i loses the game (and his opponent wins) if it becomes his turn to move from a vertex v such that no arc (v, x) is in A_i.

If $A_1 = A_2$, the game is called "impartial" or "symmetric;" otherwise it is called "partial" or "asymmetric."† Most of the literature on games of this sort has concentrated on symmetric games,‡ but we will show that the vertices of an asymmetric game can be characterized in a natural way, which generalizes the results obtained for symmetric games [2].

In particular, we will characterize the set W_i of vertices that are *winning* for player i in the sense that if he chooses a vertex w in W_i then he can assure himself of eventually winning the game. Similarly, we will find the set L_i of vertices which are *losing* for player i, such that if he chooses a vertex v in L_i then he cannot prevent an eventual loss, and the set of *drawing* vertices D_i which are neither winning nor losing. Since the game is asymmetric, the resulting partition of the vertices is in general dependent on which of the two players is under consideration. Following Steinhaus [3] and Smith [4], we will also be interested in characterizing the (minimax) number of moves which remain from each vertex. (Steinhaus' interest in this question arose from the consideration of problems of naval pursuit.)

It will be convenient to consider, for every vertex x, the set of vertices from which player i can reach x, defined by $R_i(x) = \{v \in V | (v,x) \in A_i\}$. For every set of vertices S, denote the set

*The game starts when player 1 selects a vertex from some initial set I of vertices.
†An equivalent model (cf. Smith [4]) treats asymmetric games as having one set of arcs, but two sets of vertices.
‡With the notable exception of a recent book by J. H. Conway [1].

of vertices from which player i can reach some vertex in S by $R_i(S) = \bigcup_{x \in S} R_i(x)$, and let $U_i(S) = V - R_i(S)$ be the set of vertices from which the set S is unreachable by player i.

Note that $U_i(V)$ is the set of vertices from which player i can reach no other vertex: i.e. the set of vertices from which player i has no permissible moves. So player j ($j \neq i$) wins the game if he chooses a vertex v in $U_i(V)$.

For each positive integer n, define the sets of vertices B_n and C_n by $B_0 = C_0 = \phi$, $B_n = U_2(U_1(B_{n-1}))$, and $C_n = U_1(U_2(C_{n-1}))$. Observe that $C_1 = U_1(U_2(\phi)) = U_1(V)$, and $B_1 = U_2(V)$. So player 2 wins if he chooses a vertex in C_1, and player 1 wins by choosing a vertex in B_1.

The relationship between the sets B_n and C_n is given, for every n, by the following proposition:

PROPOSITION: (i) $B_{n-1} \subset B_n$, and $C_{n-1} \subset C_n$. (ii) $B_n \subset U_2(C_n)$, and $C_n \subset U_1(B_n)$.

PROOF: Observe that for $S, T, \subset V$, if $S \subset T$ then $U_i(S) \supset U_i(T)$, and $U_i(U_j(S)) \subset U_i(U_j(T))$, for $\{i, j\} = \{1, 2\}$. Also note that, when $n = 1$, both propositions (i) and (ii) hold (since for any $S \subset V$, $\phi \subset S$ and $U_i(V) \subset U_i(S)$).

Suppose that for some fixed n, it has been shown that $B_{n-1} \subset B_n$ and $C_{n-1} \subset C_n$. Then $B_n = U_2(U_1(B_{n-1})) \subset U_2(U_1(B_n)) = B_{n+1}$, and $C_n = U_1(U_2(C_{n-1})) \subset U_1(U_2(C_n)) = C_{n+1}$, and so proposition (i) is true for all n.

Suppose it has been shown for some n that $B_n \subset U_2(C_n)$ and $C_n \subset U_1(B_n)$. Then $U_1(B_n) \supset U_1(U_2(C_n)) = C_{n+1}$, and so $B_{n+1} = U_2(U_1(B_n)) \subset U_2(C_{n+1})$. The same argument shows that $C_{n+1} \subset U_1(B_{n+1})$, and so proposition (ii) is true for all n.

Note that proposition (i) together with the facts that V is finite implies that there is some integer k such that $B_k = B_{k+1} = U_2(U_1(B_k))$, and $C_k = C_{k+1} = U_1(U_2(C_k))$. Of course proposition (i) also implies that $B_k = \bigcup_n B_n \equiv B$ and $C_k = \bigcup_n C_n \equiv C$.

We can now characterize the sets of vertices which are winning, losing, and drawing for each player, as follows:

THEOREM: (i) $W_1 = B$, and $W_2 = C$. (ii) $L_1 = R_2(C)$, and $L_2 = R_1(B)$. (iii) $D_1 = U_2(C) - B$, and $D_2 = U_1(B) - C$.

PROOF: First we show that W_1 contains B; i.e. if player 1 picks a vertex in B, then he can make his subsequent choices so as to eventually win the game. Recall that $B = \bigcup_n B_n$, and suppose that player 1 picks a vertex b_n in B_n. If b_n is in B_1 then player 1 has won, otherwise player 2 picks a vertex x such that $b_n \in R_2(x)$. Since $B_n = U_2(U_1(B_{n-1}))$, it follows that $b_n \in R_2(U_1(B_{n-1}))$, and hence $x \in U_1(B_{n-1})$: i.e. $x \in R_1(B_{n-1})$. So no matter what vertex x is picked by player 2, player 1 will always be able to respond by choosing a vertex b_{n-1} in B_{n-1}. After at most n choices of this sort, player 1 picks a vertex b_1 in B_1, and wins. So W_1 contains B, and similar argument shows that W_2 contains C.

It is an immediate consequence that L_1 contains $R_2(C)$ and L_2 contains $R_1(B)$, since if player 2, for instance, picks a vertex in $R_1(B)$, then he cannot prevent player 1 from picking a vertex in B and eventually winning. It only remains to show that $D_1 = U_2(C) - B$ and $D_2 = U_1(B) - C$: this will exhaust the set of vertices, and thus prove parts (i) and (ii), as well as (iii).

Suppose player 2 picks a vertex $v \in U_1(B) - C$. Then player 1 must choose a vertex w such that $v \in R_1(w)$, and so $w \in B$, since $v \in U_1(B)$. If player 1 chooses a vertex $w \in R_2(C)$, then we have seen that he cannot prevent his eventual loss. However, we observe that player 1 can always choose a vertex $w \in U_2(C)$, since $v \in C = U_1(U_2(C))$; i.e. since $v \in R_1(U_2(C))$. Thus, whenever player 2 picks a vertex in $U_1(B) - C$, player 1 can always respond by choosing a vertex in $U_2(C) - B$ (and his only other choice is to choose a vertex in $R_2(C)$).

Similarly, whenever player 1 chooses a vertex in $U_2(C) - B$, player 2 cannot reach a vertex in C, but he can respond by choosing a vertex in $U_1(B) - C$. So $D_1 = U_2(C) - B$ and $D_2 = U_1(B) - C$: if player i picks a point in D_i, player j ($j \neq i$) can always choose a point in D_j, and must choose such a response to avoid an eventual loss. This completes the proof of the theorem.

As a final note, observe that if player 1, say, picks a winning vertex v in B, then he can assure a win after making m more choices, where m is the number such that $v \in A_{m+1} - A_m$. If player 1 chooses a losing vertex $v \in R_2(C)$, then he can count on making only p moves before losing, where p is the number such that $v \in R_2(C_{p+1}) - R_2(C_p)$.

For example, consider the graph with vertices $V = \{a, b, c, d, e, f\}$ and arcs $A_1 = A' \cup \{(d, e)\ (e, e)\}$ and $A_2 = A' \cup \{(c, f), (f, f)\}$, where $A' = \{(a, d), (d, c), (c, b), (b, a)\}$. Then it is straightforward to verify that $W_1 = B_2 = \{a, e\}$, $W_2 = C_1 = \{f\}$, $L_1 = \{c, f\}$, $L_2 = \{b, d, e\}$, $D_1 = \{b, d\}$, and $D_2 = \{a, c\}$. Thus, for instance, if player 1 moves to vertex b he can assure himself of a draw, but if player 2 moved to b, he could not prevent a loss.

Of course, in a symmetric game this could not occur: a vertex which is losing for one player would also be losing for the other. For a discussion of the symmetric case, and its relationship to classical concepts of stability in games and graphs, see Roth [2].

REFERENCES

[1] Conway, J. H., *On Numbers and Games*, Academic Press, New York, (1976).

[2] Roth, Alvin E., "Two Person Games on Graphs," mimeo, (1975). Also Journal of Combinatorial Theory, B, *24*, 2, (1978).

[3] Steinhaus, H., "Definitions for a Theory of Games and Pursuit," Myśl. Akademicka, Lwów, *1*, 13-14 (1925). Reprinted in Naval Research logistics Quarterly, *7*, 105-108 (1960).

[4] Smith, C. A. B., "Graphs and Composite Games" Journal of Combinatorial Theory, *1*, (1) (1966).