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Two-Person Games on Graphs

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Every vertex of an abstract-directed graph is characterized in terms of a two-person game. A vertex is *winning* if by choosing it a player can assure himself of a win, it is *losing* if by choosing it he cannot prevent his opponent from winning, and it is *drawing* if it is neither winning nor losing. The sets of winning, losing, and drawing vertices are identified in terms of a set-valued function on the graph.

INTRODUCTION

Consider a game played by two players on a graph (V, D), where V is a (finite) set of vertices and $D \subset V \times V$ is a set of directed arcs. The set of *predecessors* of a vertex $v \in V$ is the set $P(v) = \{x \in V \mid (x, v) \in D\}$ and the set of predecessors of a subset $S \subset V$ is the set $P(S) = \bigcup_{v \in S} P(v)$. For each $S \subset V$ the complement of $P(S)$ is denoted $U(S) = V - P(S)$. Thus $U(S)$ is the set of vertices which do not precede any vertex in S. Denote by U^2 the composition of U with itself.

The game is played as follows: Player I selects a vertex $v_1 \in V_1 \subset V$. Player II selects any vertex v_2 such that $v_1 \in P(v_2)$. Player I then selects any vertex v_3 such that $v_2 \in P(v_3)$, and so forth. If a player selects a vertex v such that $v \in U(V)$, i.e., a vertex which precedes no other, then that player wins and his opponent losses. We call the set $U(V)$ the set of *terminal vertices*.

A set $S \subset V$ such that $S = U(S)$ is called a *kernel* of the graph. It is well known (cf. [1, p. 320]) that if the graph possesses a kernel S, and if a player chooses a vertex in S, then he can play in such a way that he is assured of a win or a draw; i.e., he can assure that he will not lose.

In this paper we characterize all vertices of the graph. A vertex is *winning* if by choosing it a player can assure himself of a win, *losing* if by choosing it he cannot prevent his opponent from winning, and *drawing* if it is neither winning nor losing. We also determine for each winning vertex the minimum number of moves in which a player can assure a win, and for each losing vertex the maximum number of moves in which a player can forestall a loss.

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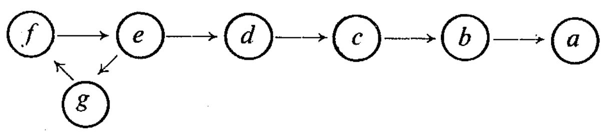
Thus if a player chooses a vertex $v \in U(A_{n^*}) - A_{n^*}$, his opponent can respond only by choosing a vertex w which is not in A_{n^*}. If $w \in P(A_{n^*})$, then the opponent cannot prevent a loss. But we have shown that for each such vertex v, the opponent can choose another vertex $x \in U(A_{n^*}) - A_{n^*}$. Thus, once a vertex v in $U(A_{n^*}) - A_{n^*}$ is chosen, each player can choose in such a way that he never chooses a vertex in $P(A_{n^*})$, and his opponent never has an opportunity to choose a vertex in A_{n^*}. In particular, his opponent never has an opportunity to choose a terminal vertex. So each player can make his choices in such a way as to assure that he will not lose, and his opponent will not win. Hence no vertex in $U(A_{n^*}) - A_{n^*}$ is either winning or losing.

It is clear from the proof that if a player chooses a winning vertex v, then the minimum number of choices in which he can be sure that he will win is equal to the smallest m such that $v \in A_m$. If a player chooses a losing vertex v, then the maximum number of choices which he can be sure of making before he loses is equal to the smallest m such that $v \in P(A_m)$. In the terminology of Smith, a vertex v in the set $A_n - A_{n-1}$ has a *remoteness* of $2(n - 1)$ from the terminal nodes—he is referring to the total number of choices remaining to both players in an optimal play of the game.

EXAMPLE.



In this graph, which has no kernel, the sets $A_1 = \{a\}$; $A_2 = A_n{}^* = \{a, c\}$; $P(A_n{}^*) = \{b, d\}$; and $U(A_n{}^*) - A_n{}^* = \{e, f, g\}$.

Concluding Remarks

The assumption that V is a finite set is made only to simplify the presentation. If V is a set of arbitrary ordinality, we must define whether a vertex is "winning" if it results in the choice of a terminal vertex after only a finite or after an arbitrary sequence of choices. In either case, generalization of the results presented here is straightforward (cf. [4]).

It is not difficult to show that any kernel of a graph must contain every winning vertex, or that if a graph has no circuits every vertex is either winning or losing, since in this case $A_n{}^* = U(A_n{}^*)$ is the unique kernel (cf. [1, p. 311]). (The set $A_n{}^*$ is the smallest fixed-point of the kind defined in [3], and every kernel is a maximal fixed-point of the same kind.)

Like the kernel, which has the same mathematical structure as a *solution* of a cooperative game (cf. [6], the set $A_n{}^*$ has the same structure as the *supercore* of a cooperative game (cf. [4]).