Two-Person Games on Graphs

ALVIN E. ROTH

University of Illinois, Urbana, Illinois 61801

Communicated by the Managing Editors

Received September 8, 1975

Every vertex of an abstract-directed graph is characterized in terms of a two-
person game. A vertex is winning if by choosing it a player can assure himself
of a win, it is losing if by choosing it he cannot prevent his opponent from winning,
and it is drawing if it is neither winning nor losing. The sets of winning, losing,
and drawing vertices are identified in terms of a set-valued function on the

INTRODUCTION

Consider a game played by two players on a graph \((V, D)\), where \(V\) is a
(finite) set of vertices and \(D \subseteq V \times V\) is a set of directed arcs. The set of
predecessors of a vertex \(v \in V\) is the set \(P(v) = \{x \in V \mid (x, v) \in D\}\) and the set
of predecessors of a subset \(S \subseteq V\) is the set \(P(S) = \bigcup_{v \in S} P(v)\). For each
\(S \subseteq V\) the complement of \(P(S)\) is denoted \(U(S) = V - P(S)\). Thus \(U(S)\) is the
set of vertices which do not precede any vertex in \(S\). Denote by \(U^2\) the
composition of \(U\) with itself.

The game is played as follows: Player I selects a vertex \(v_1 \in V_1 \subseteq V\). Player II
selects any vertex \(v_2\) such that \(v_1 \in P(v_2)\). Player I then selects any vertex \(v_3\)
such that \(v_2 \in P(v_3)\), and so forth. If a player selects a vertex \(v\) such that
\(v \in U(V)\), i.e., a vertex which precedes no other, then that player wins and
his opponent losses. We call the set \(U(V)\) the set of terminal vertices.

A set \(S \subseteq V\) such that \(S = U(S)\) is called a kernel of the graph. It is well
known (cf. [1, p. 320]) that if the graph possesses a kernel \(S\), and if a player
chooses a vertex in \(S\), then he can play in such a way that he is assured of a win
or a draw; i.e., he can assure that he will not lose.

In this paper we characterize all vertices of the graph. A vertex is winning
if by choosing it a player can assure himself of a win, losing if by choosing
it he cannot prevent his opponent from winning, and drawing if it is neither
winning nor losing. We also determine for each winning vertex the minimum
number of moves in which a player can assure a win, and for each losing
vertex the maximum number of moves in which a player can forestall a loss.
Thus if a player chooses a vertex \( v \in U(A_{n^*}) \setminus A_{n^*} \), his opponent can respond only by choosing a vertex \( w \) which is not in \( A_{n^*} \). If \( w \in P(A_{n^*}) \), then the opponent cannot prevent a loss. But we have shown that for each such vertex \( v \), the opponent can choose another vertex \( x \in U(A_{n^*}) \setminus A_{n^*} \). Thus, once a vertex \( v \) in \( U(A_{n^*}) \setminus A_{n^*} \) is chosen, each player can choose in such a way that he never chooses a vertex in \( P(A_{n^*}) \), and his opponent never has an opportunity to choose a vertex in \( A_{n^*} \). In particular, his opponent never has an opportunity to choose a terminal vertex. So each player can make his choices in such a way as to assure that he will not lose, and his opponent will not win. Hence no vertex in \( U(A_{n^*}) \setminus A_{n^*} \) is either winning or losing.

It is clear from the proof that if a player chooses a winning vertex \( v \), then the minimum number of choices in which he can be sure that he will win is equal to the smallest \( m \) such that \( v \in A_m \). If a player chooses a losing vertex \( v \), then the maximum number of choices which he can be sure of making before he loses is equal to the smallest \( m \) such that \( v \in P(A_m) \). In the terminology of Smith, a vertex \( v \) in the set \( A_n \setminus A_{n-1} \) has a remoteness of \( 2(n-1) \) from the terminal nodes—he is referring to the total number of choices remaining to both players in an optimal play of the game.

**Example.**

![Graph diagram](insert_graph_diagram)

In this graph, which has no kernel, the sets \( A_1 = \{a\} \); \( A_2 = A_{n^*} = \{a, c\} \); \( P(A_{n^*}) = \{b, d\} \); and \( U(A_{n^*}) \setminus A_{n^*} = \{e, f, g\} \).

**Concluding Remarks**

The assumption that \( V \) is a finite set is made only to simplify the presentation. If \( V \) is a set of arbitrary ordinality, we must define whether a vertex is “winning” if it results in the choice of a terminal vertex after only a finite or after an arbitrary sequence of choices. In either case, generalization of the results presented here is straightforward (cf. [4]).

It is not difficult to show that any kernel of a graph must contain every winning vertex, or that if a graph has no circuits every vertex is either winning or losing, since in this case \( A_{n^*} = U(A_{n^*}) \) is the unique kernel (cf. [1, p. 311]). (The set \( A_{n^*} \) is the smallest fixed-point of the kind defined in [3], and every kernel is a maximal fixed-point of the same kind.)

Like the kernel, which has the same mathematical structure as a solution of a cooperative game (cf. [6], the set \( A_{n^*} \) has the same structure as the supercore of a cooperative game (cf. [4]).