

A NOTE ON VALUES AND MULTILINEAR EXTENSIONS

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ABSTRACT

An axiomatic formulation is given of a class of values for cooperative games. This class includes the Shapley value and the Banzhaf index, and is related to the multilinear extension of a game.

In a recent paper [8], Owen extends the definition of the Banzhaf* index [1, 2] to non-simple games. He characterises it in terms of the multilinear extension of games, which permits an intriguing comparison with the characterization of the Shapley value which he obtained in [7].

Here we will develop an axiomatic formulation of this index which will permit comparisons to be made with the original formulation of the Shapley value [12], with the recent interpretations of the Shapley value as a cardinal utility [11], and with recent treatments of the Banzhaf and related indices for simple games [3, 5, 6, 10]. We will also characterize a wide class of related values.

For a set of players $N = \{1, \dots, n\}$, a game in characteristic function form is a real valued function on the subsets of N , such that $v(\Phi) = 0$. For a given game v , a coalition $T \subset N$ is a carrier if for all $S \subset N$, $v(S) = v(T \cap S)$. If π is a permutation of N , then for all $S \subset N$ we denote the image of S under π by πS , and define the game πv by $\pi v(\pi S) = v(S)$.

For each player $i \in N$ the *extended † Banzhaf-Coleman index* for a game v is defined by

$$(1) \quad \psi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{1}{2^{n-1}} [v(S) - v(S-i)]$$

Thus for each player i in a game v , the extended Banzhaf-Coleman index is a weighted sum of the contributions which player i makes to each coalition of which he is a member.

The sum over all players of such contributions will be denoted by

$$T(v) = \sum_{i \in N} \sum_{i \in S} [v(S) - v(S-i)]$$

*The definitions given by Coleman in [4] are also closely related.

†We use the term "extended" to denote the fact that the original definitions of Banzhaf and Coleman applied only to simple games, while equation (1) extends the definition to non-simple games.

We will show that $\psi = (\psi_1, \dots, \psi_n)$ is the unique function $f = (f_1, \dots, f_n)$ which satisfies the following conditions. (Note the similarity of these conditions both to the axioms which define the Shapley value [12], and to the axioms which determine the Banzhaf-Coleman index for simple games) [5].

CONDITION 1: (Symmetry) For each permutation π , $f_{\pi i}(\pi v) = f_i(v)$.

CONDITION 2: (Additivity) For any games v and w , $f(v+w) = f(v) + f(w)$.

CONDITION 3: For each carrier S of v , $\sum_{i \in S} f_i(v) = \frac{T(v)}{2^{n-1}}$

THEOREM 1: The extended Banzhaf-Coleman index is the unique function which satisfies Conditions 1, 2, and 3.

For each $R \subset N$, we will find it convenient to consider games of the form cv_R defined by $cv_R(S) = \begin{cases} c & \text{if } R \subset S \\ 0 & \text{otherwise.} \end{cases}$ In general we will denote the cardinality of sets R, S, T by r, s, t .

LEMMA 1: If f obeys conditions 1 and 3, then

$$f_i(cv_R) = \begin{cases} c/2^{r-1} & \text{if } i \in R \\ 0 & \text{if } i \notin R \end{cases}$$

PROOF: Suppose $i \notin R$. Then R and $R \cup i$ are both carriers of the game v_R , so Condition 3 implies $f_i(cv_R) = 0$. But if $i, j \in R$, then Condition 1 implies $f_i(cv_R) = f_j(cv_R)$. So for $i \in R$,

$$f_i(cv_R) = \frac{1}{r} \frac{T(cv_R)}{2^{n-1}} = \frac{c2^{n-r}}{2^{n-1}} = \frac{c}{2^{r-1}}.$$

PROOF OF THEOREM: Every game v is a sum of games of the form cv_R . In fact (cf.[12]), we can write $v = \sum_{R \subset N} c_R v_R$ where

$$(2) \quad c_R = \sum_{T \subset R} (-1)^{r-t} v(T).$$

Thus if f is a function which obeys Conditions 1, 2, and 3, then Lemma 1 and Condition 2 imply that f is given by

$$f_i(v) = \sum_{R \subset N} f_i(c_R v_R) = \sum_{\substack{R \subset N \\ i \in R}} c_R \frac{1}{2^{r-1}}.$$

So by equation (2), we have

$$f_i(v) = \sum_{\substack{R \subset N \\ i \in R}} \sum_{T \subset R} (-1)^{r-t} v(T) \frac{1}{2^{r-1}} = \sum_{T \subset N} \left(\sum_{\substack{R \subset N \\ R \supset \{T \cup i\}}} (-1)^{r-t} \frac{1}{2^{r-1}} \right) v(T).$$

If we denote the term in parentheses by $g_i(T)$, then

$$(3) \quad f_i(v) = \sum_{\substack{T \subset N \\ i \in T}} g_i(T) [v(T) - v(T-i)].$$

But, since $i \in T$, there are $\binom{n-t}{r-t}$ coalitions R of size r such that $R \supset \{T \cup i\}$, so

$$g_i(T) = \sum_{r=t}^n (-1)^{r-t} \binom{n-t}{r-t} \frac{1}{2^{r-1}}.$$

Letting $q = r - t$ and $m = n - t$ we have

$$g_i(T) = \sum_{q=0}^m (-1)^q \binom{m}{q} \frac{1}{2^{q+t-1}}.$$

So by the binomial theorem,

$$g_i(T) = \frac{1}{2^{i-1}} \left(1 - \frac{1}{2}\right)^{n-i} = \frac{1}{2^{n-i}}.$$

Substituting this into equation (3), we see that it is the same as equation (1). Thus equation (1) gives the only function which might obey Conditions 1, 2, and 3. It is easy to confirm that ψ obeys the conditions, which completes the proof.

It should be noted that the conditions which determine the extended Banzhaf-Coleman index are very similar to those which define the Shapley value. If the right-hand side of Condition 3 were changed from $T(v)/2^{n-1}$ to $v(T)$, then the modified conditions would define the Shapley value, given by the familiar formula

$$\phi_i(v) = \sum_{S \subset N} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S-i)].$$

Thus both the Shapley value and the extended Banzhaf-Coleman index obey Conditions 1 and 2, and both are weighted sums of the terms $[v(S) - v(S-i)]$. Both functions also obey the following, weaker version of Condition 3.

CONDITION 3a: For any carriers T, S , of v , $\sum_{i \in T} f_i(v) = \sum_{i \in S} f_i(v)$.

In general, we can state the following theorem.

THEOREM 2: A function $f = (f_1, \dots, f_n)$ obeys Conditions 1, 2, and 3a if and only if

$$(4) \quad f_i(v) = \sum_{\substack{T \subset N \\ i \in T}} k_i(t) [v(T) - v(T-i)]$$

where k_i is a real valued function on $t=1, \dots, n$.

PROOF: Clearly a function f given by equation (4) satisfies Conditions 1 and 2. It also satisfies Condition 3a, since if S is a carrier for v and $j \notin S$, then $f_j(v) = 0$. So if \hat{S} is the minimum carrier of v , then $\sum_{i \in S} f_i(v) = \sum_{i \in \hat{S}} f_i(v)$.

If f obeys Conditions 1, 2, and 3a then, following the proof of Theorem 1, we see that $f_i(v_R)$ depends only on r . So f can be written in the form of equation (4), where

$$k_i(t) = \sum_{r=t}^n (-1)^{r-t} \binom{n-t}{r-t} \hat{f}_i(r) \text{ and } \hat{f}_i(r) = f_i(v_R) \text{ for } i \in R.$$

A useful tool for examining other weighted sums is the multilinear extension of a game, as defined in [7]. The multilinear extension of a game v is defined on the unit cube of dimension n by

$$f(x_1, \dots, x_n) = \sum_{S \subset N} \left\{ \prod_{i \in S} x_i \prod_{i \notin S} (1-x_i) \right\} v(S).$$

Note that if (x_1, \dots, x_n) is a vector of zeroes and ones, then $f(x) = v(S)$ where $i \in S$ if and only if $x_i = 1$.

Differentiating with respect to x_i , we have

$$f_i(x_1, \dots, x_n) = \sum_{\substack{S \subset N \\ i \in S}} \left\{ \prod_{\substack{j \in S \\ j \neq i}} x_j \prod_{j \notin S} (1-x_j) \right\} [v(S) - v(S-i)].$$

Owen observed in [8] that the extended Banzhaf-Coleman index is obtained by evaluating the gradient of f at the point $(\frac{1}{2}, \dots, \frac{1}{2})$. In fact, we see that at every point the gradient of f is a weighted sum of terms of the form $[v(S) - v(S-i)]$. So by Theorem 2, we have the following result.

COROLLARY 1: The gradient of the multilinear extension of a game everywhere satisfies Conditions 1, 2, and 3a.

In [11] it was shown that the Shapley value is a von Neumann-Morgenstern utility function, characterized by the fact that $\phi_i(v_R) = \frac{1}{r}$ for $i \in R$. It can be shown that every function f which satisfies Conditions 1, 2, and 3a is also a utility, characterized by the numbers $f_i(v_R)$. The author hopes to explore this further in a subsequent article [9].

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