The Shapley Value as Applied to Cost Allocation: A Reinterpretation

ALVIN E. ROTH AND ROBERT E. VERRECCHIA*

1. Introduction

The problem of cost allocation is inescapable in virtually every organization and consequently pervades every facet of accounting. As an alternative to traditional accounting allocation bases, there is a growing interest in cost allocation schemes predicated on notions in game theory. Shubik [1962] was among the original proponents of this. He suggested the Shapley value as a method of joint-cost allocation, and it is the Shapley value that has continued to attract the widest interest.[1]

The Shapley value was introduced by Shapley [1953] as a method for each player to assess a priori the benefits he would expect from playing a game. To show its application to the problem of assigning joint cost, let us suppose that the full cost of some common service (e.g., a computer facility, a power plant, or a maintenance staff) is to be shared among n departments, which will be designated by $N = \{1, 2, \cdots, n\}$. The function $v(S)$ describes the net total benefit to the coalition S when those departments cooperate to secure the common service. For purposes of this discussion, it will be assumed that the total net benefit is expressed in

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* Associate Professor and Assistant Professor, University of Illinois at Urbana-Champaign. [Accepted for publication July 1978.]

[1] Recent papers dealing with the Shapley value as a method of cost allocation include Loehman and Whinston [1971; 1974], Champsaur [1975], Littlechild and Thompson [1977], Hamlen, Hamlen, and Tschirhart [1977], and Jensen [1977]. In addition, a manuscript by Arthur L. Thomas which will include a discussion of the Shapley value is in preparation but has not, as yet, been received. At least one cost allocation scheme based on the Shapley value has actually been implemented to allocate costs among users of a telephone system (see Billera, Heath, and Raanan [1978]). It has been pointed out that a potential disadvantage of the Shapley value is the computational burden. However, Megiddo [1978], in considering a particular class of cost allocation games, shows that efficient algorithms for computing the Shapley value do exist.

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terms of income and therefore can be transferred between departments when they make "side payments" to one another. The Shapley value for department i is:

$$b_i = \sum_{S \subset N} \frac{(s-1)! \, (n-s)!}{n!} [v(S) - v(S - \{i\})],$$

where s is the number of departments in coalition S, and n is the total number of departments. The charge for the common service can then be assigned by charging department i its gross benefit less its Shapley value, b_i. That is, $b_i = B_i - C_i$ where B_i is the gross benefit to i and C_i is the charge to i. Thus $C_i = B_i - b_i$.

The intuitive explanation for the Shapley value has been that it is the expected marginal benefit added by each department if all orderings of departments are equally likely. That is, the Shapley value can be computed by calculating the average marginal benefit which department i brings to a coalition S, under the assumption that coalitions form in random order. Thus, there are $(s-1)! \, (n-s)!$ orderings of the departments, such that department i comes after all the other departments in a given coalition S (which contains i), but before any department which is not in the coalition S. In this case, the marginal contribution of department i is $v(S) - v(S - \{i\})$. Since there are $n!$ different orderings of the departments, the expected marginal contribution of department i is the sum of its marginal contributions to each coalition S, each weighted by the proportion of the orderings in which the arrival of department i forms that coalition. Unfortunately, this explanation is not entirely appealing, since, in general, there is no reason that the formation of coalitions should be regarded as equally likely events. In any event, it says nothing about the accounting objectives of fairness, equity, and neutrality.

Shapley's original formulation of the problem of defining a value for games characterized the Shapley value as the unique function obeying a certain set of axioms. However, these axioms have proved difficult to interpret in a compelling way from the point of view of cost allocation (e.g., see Shubik [1962, p. 335] and Spinetto [1975, p. 486]). This paper will attempt to offer a new interpretation of the Shapley value as it applies to the problem of cost allocation. Our discussion relies on some recent results in game theory which will help us identify the circumstances in which the Shapley value provides an appropriate cost allocation mechanism, and those circumstances in which it does not. In particular, our interpretation will suggest that the Shapley value may provide a costless surrogate for allowing the cost allocation to be determined by bargaining among interested parties. Furthermore, the surrogate is consistent with the objectives of fairness, equity, and neutrality suggested by accounting theory.

2. Determining Managers' Expected Utility for Bargaining

2.1 INTRODUCTION: THE ROLE OF BARGAINING IN THE ASSIGNMENT OF JOINT COST

Joint costing arises because departments, or other reasonably cohesive subunits of an organization such as "product-lines," collectively share common services, possibly in the form of materials, labor, or organizational skill. Suppose that a firm delegated the responsibility of assigning joint costs to the departments themselves. Then the only way common costs could be allocated would be for representatives of the various subunits (e.g., managers or department heads) to bargain among themselves. That is, they would bargain about what (if any) jointly shared services should be provided and how the cost should be shared.

Of course, the firm would experience unnecessary costs if it made all of its service acquisition and cost allocation decisions by permitting its managers to bargain with each other in an unrestricted fashion. Presumably, the firm can organize departments in a manner that will reduce, or eliminate, those costs which arise through bargaining, not the least of which is the risk of reaching a decision which is suboptimal from the perspective of the firm as a whole. In addition, there is a real expenditure in time and energy incurred by managers, in bargaining, which could otherwise be more usefully employed in guiding the internal affairs of their departments.

One way to effect a "bargained" cost allocation scheme without incurring these attendant costs is for the firm to implement a priori the cost assignment such that if individual bargaining costs were zero,[2] each manager would be indifferent between receiving the cost assignment for certain or actually bargaining toward an uncertain outcome. The outcome of a bargaining process is uncertain because it depends on the strategic interaction of rational managers. Although considerable literature has been devoted to the study of making choices which involve probabilistic uncertainty, decisions involving strategic uncertainty cannot be completely described by probability distributions. Recent work in game theory, however, has suggested that, depending upon managers' attitudes toward risk, their expected utility for bargaining can be determined in a fashion that is equivalent to evaluating the expected utility of a (probabilistically) uncertain event. Therefore, if a firm can evaluate each manager's expected utility for bargaining, it can use this figure as a costless surrogate for the bargaining process itself.

[2] An important point here is that if individual bargaining costs were positive, managers might regard implementing any one of an infinite set of a priori cost assignments as Pareto-superior to bargaining because this would eliminate their (expected) bargaining costs. We will suggest an a priori cost assignment such that even in the event that individual bargaining costs were zero (i.e., managers would not be inhibited from bargaining simply because of the cost to them), managers would be indifferent. See also the discussion of Assumption 3.

2.2 A GAME-THEORETIC FORMULATION OF THE PROBLEM

To determine an expected utility for bargaining, the problem must be formalized in a game-theoretic structure with appropriate definitions and assumptions. For example, a (cost allocation) *game* is defined to consist of: (1) a finite set of departments within a firm (e.g., the sales/marketing department, or the Christmas light-bulb product line). For convenience we will number the departments as $1, 2, \cdots, n$ and let $N = \{1, 2, \cdots, n\}$. (2) A *benefit structure* v which associates with each coalition of departments the potential benefits available to that coalition acting alone. There are of course $2^n - 1$ nonempty coalitions of departments, and v assigns a potential benefit to each of them.

We will also need to consider the expected utility function (in the sense of von Neumann and Morgenstern [1953]) of a participant in the game. Such a utility function is a model of rational individual preferences concerning final outcomes and lotteries between outcomes. Thus, in order to model a manager's expected utility for playing a particular game (i.e., for bargaining in terms of a particular benefit structure), we must consider his preferences over games and lotteries between games.

Define a *simple prospect* as manager i's opportunity to participate in a bargaining process determined by a benefit structure v. A simple prospect will be represented by (i, v). A manager's preference for one simple prospect over another would then be expressed as:

$$(i, v), P(i, w),$$

where P denotes preference. That is, we read the above expression as "manager i prefers to bargain under benefit structure v than under benefit structure w." A *lottery* will be defined as an opportunity which gives manager i a probability g of bargaining in a cost allocation game with a benefit structure v, and probability $(1 - g)$ for bargaining in a game with a benefit structure w. For convenience, a lottery will be represented by the notation:

$$[g(i, v); (1 - g)(i, w)].$$

Lotteries between different cost allocation games arise in a natural way. For instance, suppose that a cost allocation decision must be made at the beginning of next year, but the benefit structure which will be in effect at that time will depend on whether the sales department meets some projected target. In particular, suppose that if the target is met, the resulting benefit structure will be v, and that if the target is not met, it will be w. Suppose that g is the probability that the target will be met. Then the above lottery is the one facing manager i before it is known whether the sales department will meet its target.

We will assume that a manager has well-behaved preferences[3] over

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[3] See Fishburn [1970] or Herstein and Milnor [1953] for a complete discussion of what constitutes a "well-behaved" preference relation in the sense that it can be represented by a cardinal utility function.

games and the associated set of lotteries. In particular, any two prospects can be compared: either a manager is indifferent between them, or he prefers one to the other. A von Neumann-Morgenstern utility function is a representation of preferences in terms of a real-valued function which preserves expected value. When we say a manager's preferences are represented by a utility function u, we mean that $u(i, v) > u(i, w)$ if and only if $(i, v)P(i,w)$, and $u[g(i, v); (1 - g)(i, w)] = gu(i, v) + (1 - g)u(i, w)$, that is, the utility of a lottery is equal to its expected utility. (Note that a utility function is defined uniquely only up to an arbitrary choice of origin and scale. That is, two utility functions which differ only in origin [zero point] and scale [unit] represent the same preferences.)

2.3 NORMATIVE ASSUMPTIONS WHICH IMPLY EACH MANAGER'S EXPECTED UTILITY IS THE SHAPLEY VALUE

The purpose of this section is to introduce and discuss the three normative assumptions which will imply that each manager is indifferent between the cost allocated to him by the Shapley value of bargaining over how the cost will be allocated. The first assumption is innocuous, but its role in the formal development of the theory requires that it be explicitly stated.

Define a *null department* to be one which has no concern or effect in the cost allocation game over some common service; formally, department i is a null department in a game v if for every coalition S, $v(S \cup \{i\}) = v(S)$.

ASSUMPTION 1. A manager is indifferent between any two benefit structures in which his department is null. Formally, if department i is a null department in a bargaining process v, then managers are indifferent between the prospects (i, v) and (i, v_0), where v_0 is the benefit structure in which no coalition of departments can secure any benefits (i.e., in which there are no costs or benefits to be divided, so that every department is null). That is, a manager is indifferent between a game in which he does not participate and a game in which there are no benefits to be shared by any departments. This is a very reasonable assumption for those costs in which the benefits received by other departments have no effect on his department.

If we take the prospect (i, v_0) to be the zero point of manager i's utility function u, then Assumption 1 immediately yields the following result.

LEMMA 1. If i is a null department in the benefit structure v, then $u(i, v) = u(i, v_0) = 0$.

The next two assumptions are considerably stronger than the previous one and will be interpreted at greater length. They concern managers' attitudes toward probabilistic and "strategic" uncertainty. In particular, when a manager's attitude toward both forms of uncertainty is "neutral," his expected utility for bargaining is given by the Shapley value.

Assumption 2 is called "ordinary risk neutrality" in Roth [1977a], to suggest that it involves attitudes toward risk created by probabilistic

uncertainty. Prior to participating in a bargaining process, suppose that a manager is uncertain about which of two possible cost allocation games will be played. For instance, the needs of each department for computer services in the next year may depend on their sales this year. Consequently, a manager may associate a probability g to the possibility that a cost allocation game with a benefit structure v will be played, and $(1 - g)$ to the possibility that a cost allocation game with a benefit structure w will be played. Then Assumption 2 follows.

ASSUMPTION 2. Each manager is indifferent between being uncertain as to which cost allocation game will be played or participating for certain in a game in which the benefit structure is simply the expected benefit structure implied by the uncertain situation, that is, $gv + (1 - g)w$. Assumption 2 can be expressed formally by stating that a manager is indifferent between the lottery $[g(i, v); (1 - g)(i, w)]$ and the simple prospect $(i, gv + (1 - g)w)$. That is, we assume here that a manager is indifferent between, on the one hand, waiting until the uncertainty has resolved itself and then playing either the game v (with probability g) or the game w, or, on the other hand, playing the game in which each coalition S can obtain (for certain) its expected benefit $gv(S) + (1 - g)w(S)$.

The assumption that a manager is risk-neutral to lotteries over games appears appropriate for games in which the available benefits are defined in terms of the manager's utility function, because in this case he is risk-neutral to lotteries over benefits by definition of a utility function.[4]

If u is the utility function representing manager i's preferences, then Assumption 2 yields the following result concerning the benefit structures v, w, and $v + w$.

LEMMA 2. $u(i, v) + u(i, w) = u(i, v + w)$. The proof of this is straightforward, see Appendix A.

The counterpart to Assumption 2 is one concerning attitudes toward "strategic risk." Any bargaining process with more than one nonnull position, e.g., department or product-line, involves some potential uncertainty as to the outcome of the game. This uncertainty is called strategic risk in that it arises from the interaction of managers as representative of various departments, rather than from a process which can be described probabilistically.[5] Strategic risk will be dealt with by making the following assumption.

ASSUMPTION 3. Each manager is indifferent between bargaining among r managers for an uncertain outcome or receiving $\frac{1}{r}$ of the benefit for certain. If bargaining can be viewed as a relatively costless process for

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[4] If preferences over games are not risk-neutral to lotteries, then it is no longer possible to decompose games into components in the manner of Lemma 2. In this case, the evaluation of the resulting utility function presents unresolved technical problems.

[5] The notion of strategic risk has proved to have application to a much wider class of games than those suggested here. See, for instance, the results discussed in Roth [1977c].

each of the members of any coalition, and managers perceive bargaining positions to be identical, then Assumption 3 is most reasonable in the case in which managers perceive themselves to have equal bargaining ability. If some other assumption is made about strategic risk, expected utility can still be evaluated, but it would no longer be equal to the Shapley value (see Appendix A).

For example, consider the game in which the members of a subset R of departments can share some benefit only if they all cooperate, and in which all other departments are null (i.e., all other departments are uninvolved). This yields a benefit structure v, such that $v(S) = v(R) > 0$ if and only if the coalition S contains R, and $v(S) = 0$ otherwise. Then this game does not offer any advantage to one department in R over another. Assumption 3 requires that a manager i (in R) be indifferent between participating in this particular game or simply receiving $\dfrac{1}{r}$ of the available proceeds $v(R)$. That is, if a manager's attitude toward strategic risk is one of indifference, he expects the average benefit, since this particular game gives him no advantage or disadvantage in bargaining. If u is the utility function of manager i (where i is in R), and v_R is the game of the type described in which $v_R(R) = r$, then take $u(i, v_R) = 1$; that is, the prospect (i, v_R) defines the unit of the utility function.

The significance of these three assumptions is that they are necessary and sufficient to insure that a manager's expected utility for bargaining in a particular cost allocation game would be his Shapley value for the game. That is, under these assumptions, each manager would be indifferent between paying the cost assigned to him by the Shapley value or participating in a bargaining process. Formally, if u is a utility function, normalized as in Lemma 1 and Assumption 3, then we have the following result.

THEOREM. A manager's expected utility for playing in a game v is equal to his Shapley value if and only if his preferences obey Assumptions 1–3. (For proof, see Appendix A.)

3. Conclusion

The notions of fairness, equity, and neutrality are sufficiently broad to be subject to a variety of interpretations. Under Assumptions 1–3 of Section 2.3, the Shapley value represents managers, expected utility for bargaining, and therefore each manager would be indifferent between having his department charged its gross benefit less its Shapley value or bargaining to an uncertain outcome. In this sense, the Shapley value represents a fair, equitable, neutral, and costless surrogate for allowing managers to bargain over how costs will be allocated. Of course, this conclusion is predicated on the fact that managers' preferences obey certain assumptions. We would not expect all managers to have a neutral attitude toward both probabilistic and strategic risk in all situations. But there may be circumstances in which a firm would find it convenient to

assume that managers behaved as if their preferences obeyed these assumptions, and in these circumstances the justification for using the Shapley value as a cost assignment method would be clear. However, if a firm chose to assume otherwise, the Shapley value might not yield an entirely appropriate cost allocation scheme.

APPENDIX A

Let P be a preference relation over games, which can be represented by a von Neumann–Morgenstern utility function u, such that $u(v_0) = 0$. (For any game v we will write $u(v)$ rather than $u(i, v)$, keeping in mind that the preferences are those of a single individual, player i.)

LEMMA 2. If P obeys Assumption 2, then for any games $v, w, u(v) + u(w) = u(v + w)$.

Proof. Assumption 2 implies $(\frac{1}{2}v + \frac{1}{2}w)I[\frac{1}{2}v; \frac{1}{2}w]$, where I denotes indifference under the preference relation P. So $u(\frac{1}{2}v + \frac{1}{2}w) = u[\frac{1}{2}v; \frac{1}{2}w] = \frac{1}{2}u(v) + \frac{1}{2}u(w)$. It remains only to show that $u(\frac{1}{2}v + \frac{1}{2}w) = \frac{1}{2}u(v + w)$. This follows from the observation that, for any game z, $u(z) = \frac{1}{2}u(2z)$, since Assumption 2 implies $u(z) = u[\frac{1}{2}(2z); \frac{1}{2}v_0] = \frac{1}{2}u(2z) + \frac{1}{2}u(v_0) = \frac{1}{2}u(2z)$. Let $z = (\frac{1}{2}v + \frac{1}{2}w)$ and the lemma is proved.

THEOREM 1. If P obeys assumptions 1–3, then $u(v)$ equals the Shapley value (of player i) for the game v.

Proof. Assumptions 1 and 3 imply that u coincides with the Shapley value for games of the form v_R. Since a game is simply a sequence of numbers corresponding to each of $2^n - 1$ nonempty coalitions, the space of games is simply Euclidean space, of dimension $2^n - 1$.

But there are $2^n - 1$ distinct games of the form v_R (one for each coalition R), which can be shown to be linearly independent. Thus, the games v_R form a basis for the vector space of games. That is, every game v can be represented as a sum of games of the form v_R. By Lemma 2, the utility of playing a game is additive, as is the Shapley value. Consequently, they coincide on all games, which completes the proof.

The utility of playing a game is obviously sensitive to the risk posture expressed in Assumption 3. If the fraction $1/r$ in that assumption is replaced by an arbitrary fraction $f(r)$, then the utility of playing a game is given by:

$$u(v) = \sum_{T \subset N} k(t)[v(T) - v(T - i)], \text{ where}$$

$$k(t) = \sum_{r=t}^{n} (-1)^{r-t} \binom{n-t}{r-t} f(r).$$

Of course, when $f(r) = 1/r$, this expression simplifies to the Shapley value.

For a more complete discussion of this material, see Roth [1977b].

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