Ramsey numbers of link hypergraphs

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Joint work with Jacob Fox

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Hypergraphs

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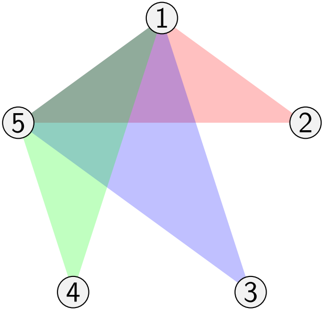

Figure: A 3-graph on 5 vertices and 3 edges.

Hypergraph Ramsey numbers

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Remark: this can be deduced from $r(K_n^{(3)}, K_n^{(3)}) < \infty$ or from $r(K_5^{(4)}, K_n^{(4)}) < \infty$.

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2. Quasirandomness conditions for hypergraphs are not all equivalent.

3. We can design Ramsey hypergraphs that are globally quasirandom but locally structured.

Diagonal Ramsey numbers

Theorem (Erdős 1947, Erdős-Szekeres 1935)

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Theorem (Erdős-Hajnal 1960s, Erdős-Rado 1952)

For $k \geq 3$,

$$t_{k-1}(\Omega(n^2)) \leq r(K_n^{(k)}, K_n^{(k)}) \leq t_k(O(n)),$$

where $t_k(n)$ is the tower function $t_1(n) = n$, $t_{k+1}(n) = 2^{t_k(n)}$.

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Remark

Both the upper and lower bounds are recursive in nature, proving bounds on uniformity $k+1$ using uniformity k. However, the lower bound (stepping up lemma) only works starting from $k = 3$.

Off-diagonal Ramsey numbers

In the graph case:

Theorem (Kim 1995, Ajtai-Komlós-Szemerédi 1980)

$$r(K_3, K_n) = \Theta\left(\frac{n^2}{\log n}\right).$$

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Central problem in the development of the probabilistic method:

- Alterations (Erdős 1961)
- Lovász Local Lemma (Spencer 1975)
- Large deviation inequalities (Krivelevich 1995)
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For almost all other H, the order of $r(H, K_n)$ is still unknown.

Off-diagonal hypergraph Ramsey numbers

Let $K_4^{(3)} - e$ be the 3-graph with 4 vertices and 3 edges.

Theorem (Erdős-Hajnal 1972)

$$2^{\Omega(n)} \leq r(K_4^{(3)} - e, K_n^{(3)}) \leq 2^{O(n \log n)}.$$

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Our main result:

Theorem (Fox-H. 2019)

$$r(K_4^{(3)} - e, K_n^{(3)}) = 2^{\Theta(n \log n)}.$$

The Erdős-Hajnal tournament construction

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Remark

A purely random 3-graph on N vertices does poorly, since edge density $p = N^{-c}$, which makes the independence number $\approx N^{c/2}$.

The Erdős-Hajnal tournament construction

Theorem (Erdős-Hajnal 1972)

$$r(K_4^{(3)} - e, K_n^{(3)}) \geq 2^{\Omega(n)}.$$

Proof.

Let T be a random tournament on $N = 2^{cn}$ vertices:

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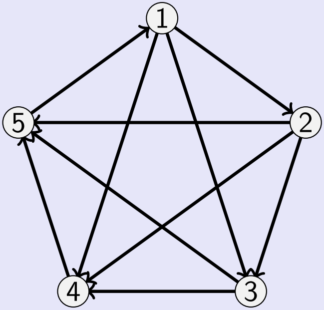

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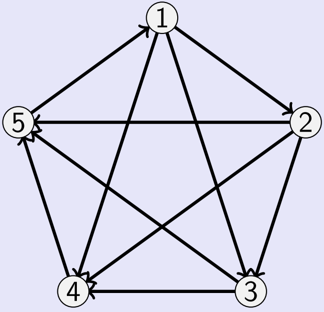

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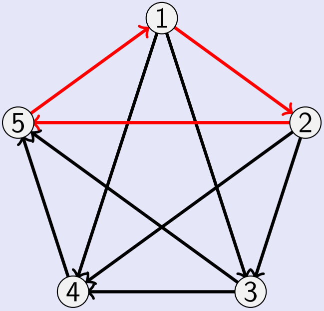

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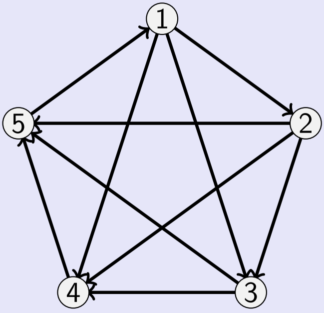

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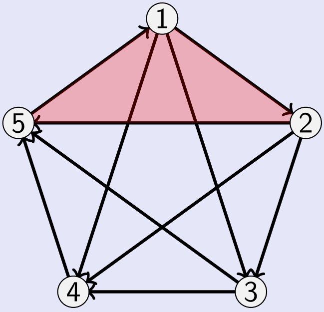

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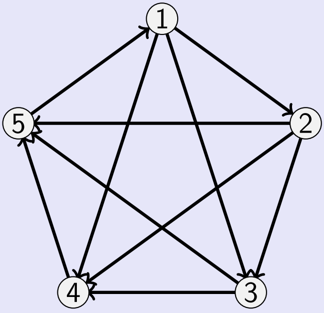

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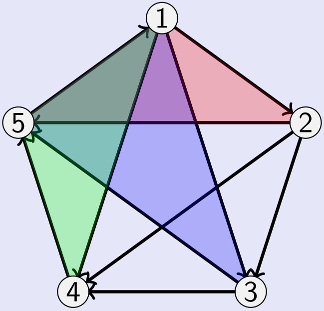

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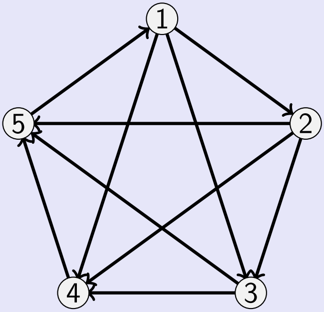

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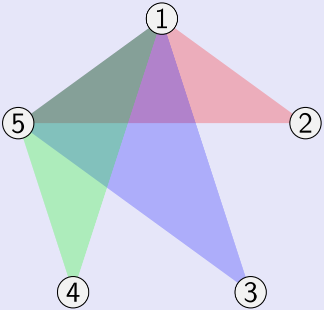

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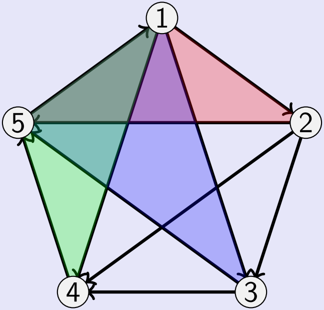
Proof (continued).

Among any four vertices in T, at most two out of four triples form cyclic triangles, so Γ doesn't contain $K_4^{(3)} - e$.



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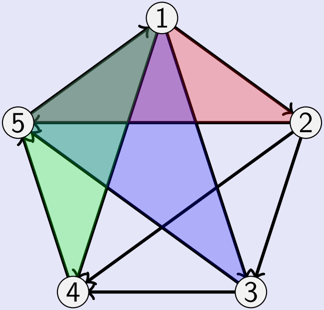
The expected number of independent sets of size n in Γ is

$$\binom{N}{n} \cdot n! \cdot 2^{-\binom{n}{2}} < 1.$$



Digression: Quasirandomness for hypergraphs

Theorem (Chung, Graham, Wilson 1989)

If H is a fixed labelled graph, G is a labelled graph on n vertices, and every vertex subset $U \subseteq V(G)$ contains $p\binom{|U|}{2} + o(n^2)$ edges, then G contains $(1 + o(1))p^{e(H)}n^{v(H)}$ labelled copies of H.

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Example: counting triangles

If every linear-sized subset of a graph G has edge density $1/4$, then G has triangle density $1/64$.

Surprise

Such a statement is false for 3-graphs!

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If Γ is the 3-graph of cyclic triangles in a random tournament T on N vertices, then every subset $U \subseteq V(G)$ contains

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- This is a type of random construction that isn't available in uniformity 2: it is quasirandom in the sense of edge densities and non-quasirandom in the sense of subgraph counts.

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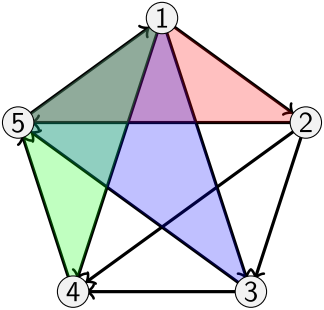
Remarks

- This is a type of random construction that isn't available in uniformity 2: it is quasirandom in the sense of edge densities and non-quasirandom in the sense of subgraph counts.
- The existence of such hypergraphs has serious implications for hypergraph regularity.

Improving the tournament lower bound

Theorem (Fox-H. 2019)

$$r(K_4^{(3)} - e, K_n^{(3)}) = 2^{\Theta(n \log n)}.$$



Links in hypergraphs

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If G is a k-graph, the *link* G_v of a vertex v in G is the $(k-1)$-graph on $V(G) \setminus \{v\}$ whose edges come from deleting v from the edges of G containing v.

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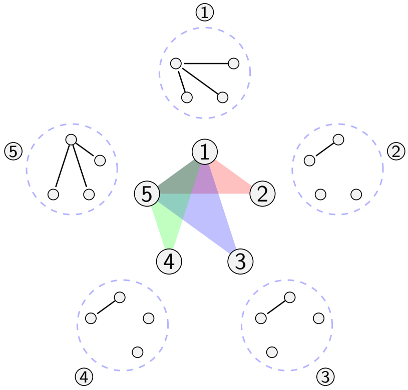

Figure: The links of each vertex.

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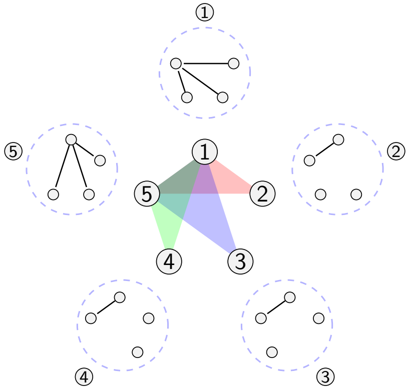
Note: G is $(K_4^{(3)} - e)$-free iff the links G_v are all triangle-free.



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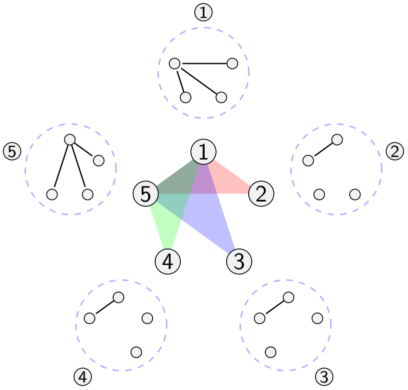

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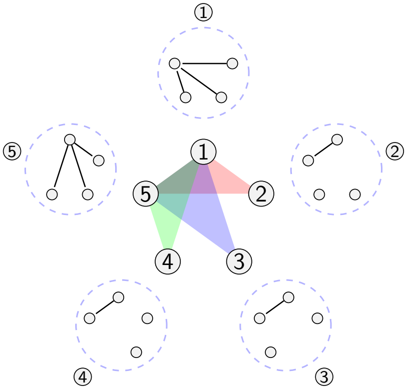

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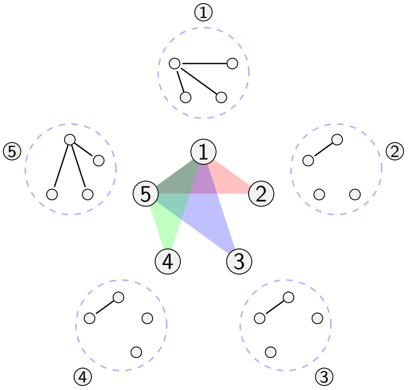
Observations

1. If Γ is the 3-graph of cyclic triangles in any tournament, then the links Γ_v of its vertices are bipartite.



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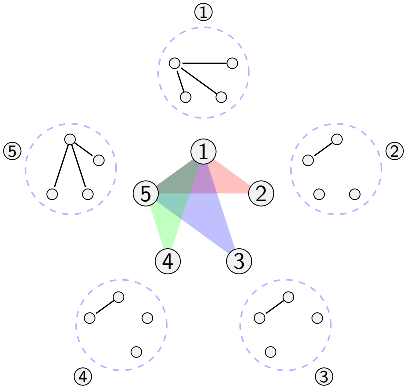

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1. If Γ is the 3-graph of cyclic triangles in any tournament, then the links Γ_v of its vertices are bipartite.
2. If Γ is *any* 3-graph with bipartite links, then Γ contains no $K_4^{(3)} - e$.
3. We can reproduce the Erdős-Hajnal lower bound by taking Γ to be a 3-graph with random bipartite links.

Random bipartite links

Modified Construction

Let Γ be a random 3-graph on N vertices specified by N bipartitions $U_v \cup W_v = V(G) \setminus \{v\}$ indexed by the vertices v. A triple $\{u, v, w\}$ is an edge of Γ iff v and w are on the opposite sides of the bipartition for u, u and w are on opposite sides of the bipartition for v, and u and v are on opposite sides of the bipartition for w.

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Lemma

If the bipartitions are chosen uniformly at random, then w.h.p. Γ *is* $(K_4^{(3)} - e)$*-free, has edge density* $1/8 + o(1)$*, and independence number* $\Theta(\log N)$*.*

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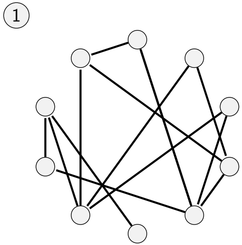
But we want independence number $n = O(\log N / \log\log N)$ to get $N = 2^{\Omega(n \log n)}$, so the links can't be bipartite.

Our Construction

Naive random construction



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Needs to be very sparse.

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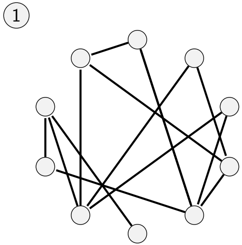

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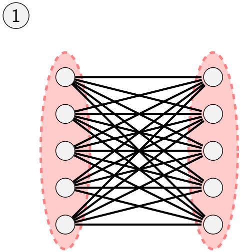

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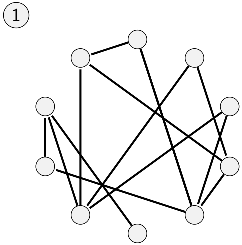

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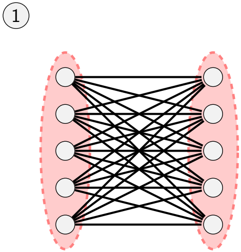
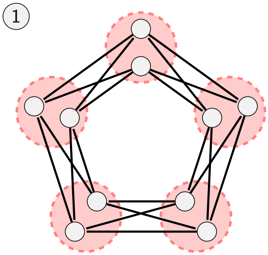


Random triangle-free links



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Our construction

Links are random blowups of a small triangle-free graph.

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Construction

Fix an auxiliary graph A on $m = n^C$ vertices, which is triangle-free and has edge density $p = m^{-2/3}$.
Let Γ be the 3-graph on $N = 2^{cn \log n}$ vertices specified by a map $\chi : V(\Gamma)^2 \to V(A)$. A triple $\{u, v, w\}$ is an edge of Γ iff $\chi(u, v) \sim \chi(u, w)$, $\chi(v, u) \sim \chi(v, w)$, and $\chi(w, u) \sim \chi(w, v)$.

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Let Γ be the 3-graph on $N = 2^{cn \log n}$ vertices specified by a map $\chi : V(\Gamma)^2 \to V(A)$. A triple $\{u, v, w\}$ is an edge of Γ iff $\chi(u, v) \sim \chi(u, w)$, $\chi(v, u) \sim \chi(v, w)$, and $\chi(w, u) \sim \chi(w, v)$.
Choose χ uniformly at random. For each v, the link Γ_v will be a subgraph of a blowup of A, so it is triangle-free.

Proof of main result

Theorem (Fox-H. 2019)

$$r(K_4^{(3)} - e, K_n^{(3)}) = 2^{\Theta(n \log n)}.$$

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Lemma (Hard part)

If Γ is the random 3-graph described above, the independence number of Γ is less than $n = O(\log N / \log \log N)$.

General link hypergraphs

Definition

If G is a k-graph, the *link hypergraph* L_G of G is the $(k+1)$-graph on $V(G) \cup \{v\}$ (for a new vertex v) whose edges come from inserting v into all the edges of G.

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Theorem (Conlon-Fox-Sudakov 2010)

If G is bipartite, then $r(L_G, K_n^{(3)}) = n^{\Theta(1)}$.
If G is non-bipartite, then $r(L_G, K_n^{(3)}) = 2^{\Omega(n)}$.

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Question

How do the implicit constants depend on G?

Diagonal Ramsey numbers via link hypergraphs

Theorem (Fox-H. 2019)

For all $s \geq 3$ and $n \geq 1$, we have

$$r(L_{K_s}, K^{(3)}_{n,n,n}) = \binom{n+s}{s}^{\Theta(n)}.$$

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Remark

This gives a very "sparse" proof that $r(K^{(3)}_n, K^{(3)}_n) = 2^{\Omega(n^2)}$, and suggests that the diagonal Ramsey number should be much bigger.

Ramsey transition thresholds

Definition

Let $r_k(s, t; n)$ be the minimum N such that in any k-graph on N vertices, either some s vertices span t edges, or else there is an independent set of size n.

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Examples

Thus $r_3(4,4;n) = r(K_4^{(3)}, K_n^{(3)})$ and $r_3(4,3;n) = r(K_4^{(3)} - e, K_n^{(3)})$.

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Conjecture (Erdős-Hajnal 1972)

For every $s > k \geq 3$, there exists a unique $t = h_1^{(k)}(s)$ such that $r_k(s,t-1;n)$ is polynomial in n and $r_k(s,t;n)$ is exponential in n.

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For every $s > k \geq 3$, there exists a unique $t = h_1^{(k)}(s)$ such that $r_k(s,t-1;n)$ is polynomial in n and $r_k(s,t;n)$ is exponential in n. In general, for every $1 \leq i \leq k-2$ there is a unique $t = h_i^{(k)}(s)$ such that $r_k(s,t-1;n)$ has tower height i and $r_k(s,t;n)$ has tower height $i+1$ (as a function of n).

The polynomial-to-exponential transition

Theorem (Conlon-Fox-Sudakov 2010)

For infinitely many s, $h_1^{(3)}(s)$ exists and $h_1^{(3)}(s) - 1 = T(s)$ which is the maximum number of cyclic triangles in a tournament on s vertices.

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Theorem (Mubayi-Razborov 2019)

For all $s > k \geq 4$, $h_1^{(k)}(s)$ exists and $h_1^{(k)}(s) - 1 = g^{(k)}(s)$ which is the maximum number of ordered rainbow tournaments on k vertices in an ordered $\binom{k}{2}$-edge-colored tournament on s vertices.

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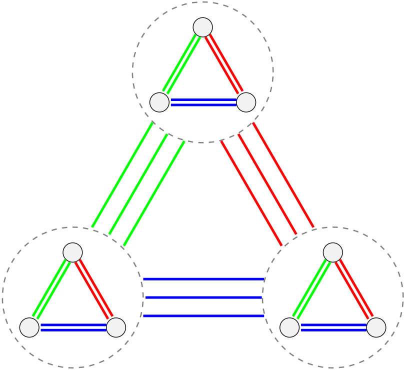
Both results relate thresholds to the general problem of *inducibility*: given a (possibly colored and/or directed) graph on k vertices, what is the maximum number of induced copies of it in a graph on s vertices?

Open problem: inducibility for $k = 3$

Conjecture (Erdős-Hajnal 1972)

For all $s \geq 3$, the maximum number of ordered rainbow triangles in a 3-edge-coloring of K_s is $g^{(3)}(s)$, where $g^{(3)}(s) = 0$ if $s < 3$ and otherwise

$$g^{(3)}(s) = \max_{a+b+c=s} \{g^{(3)}(a) + g^{(3)}(b) + g^{(3)}(c) + abc\}.$$



Skipping the exponential order

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Conjecture

For all $s \ge 4$, there exists t for which $r_3(s,t-1;n) = n^{\Theta(1)}$ and $r_3(s,t;n) = 2^{\Omega(n \log n)}$.

Thank you!