A Combinatorial Framework for Nonlinear Dynamics

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I. WHY?
Assume: there exists a multiparameter deterministic model for the dynamics \( f : X \times \Lambda \rightarrow X \) \( (X \text{ is compact}) \)

\[ f_\lambda(\cdot) = f(\cdot, \lambda) : X \rightarrow X \]

Iterations define the dynamics
Assume: there exists a multiparameter deterministic model for the dynamics \( f : X \times \Lambda \rightarrow X \) (\( X \) is compact).

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f_\lambda(\cdot) = f(\cdot, \lambda) : X \rightarrow X \quad \text{Iterations define the dynamics}
\]

Objects of Interest: Invariant sets

Bounded subsets \( S_\lambda \subset X \) such that \( f_\lambda(S_\lambda) = S_\lambda \)
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Bounded subsets \( S_\lambda \subset X \) such that \( f_\lambda(S_\lambda) = S_\lambda \)

Invariant sets are associated to asymptotic dynamics

Example: If \( f(x) = \frac{1}{2}x \) then \( S = \{0\} \)
Three Problems associated with Invariant Sets.
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1. Time series data is transient.
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2. Nonlinear systems exhibit chaos: for each parameter value there can be uncountably many topologically distinct orbits.
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\[ f(x) = rx(1 - x) \]

2. Nonlinear systems exhibit chaos: for each parameter value there can be uncountably many topologically distinct orbits.

3. Bifurcations can occur on Cantor sets of positive measure.
II. Rigorous Computational Results for Multiparameter Systems
Fundamental Decomposition:

Recurrent Dynamics vs. Gradient-like Dynamics
Fundamental Decomposition:

**Recurrent Dynamics vs. Gradient-like Dynamics**

A Morse decomposition $\mathbf{M}$ of $X$ consists of a finite poset $(\mathbf{P}, \leq)$ that labels a collection of compact disjoint invariant sets of $M(p) \subset S$, called Morse sets, such that for every $x \notin \bigcup_{p \in \mathbf{P}} M(p)$ there are indices $q < p$ in $\mathbf{P}$ such that the forward orbit of $x$ limits to $M(q)$ and the backward orbit of $x$ limits to $M(p)$. 
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The labelling by $P$ implies that a Morse decomposition can be represented as an acyclic directed graph $MG$ called the Morse graph.
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An Example

A density dependent Leslie model:

\[
\begin{align*}
1\text{st year pop. } & \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x + \theta_2 y)e^{-0.1(x+y)} \\ 0.7x \end{bmatrix} \\
2\text{nd year pop. } & \begin{bmatrix} x \\ y \end{bmatrix} \quad f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \\
& (x, y; \theta_1, \theta_2)
\end{align*}
\]
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f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2
\]

\[
(x, y; \theta_1, \theta_2)
\]

We can construct a mathematically rigorous, queryable database for the global dynamics on the phase space

\[ [0, \infty) \times [0, \infty) \]

and for all parameters

\[
\theta = (\theta_1, \theta_2) \in [8, 37] \times [3, 50]
\]
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**Input:** Nonlinear map, Phase space, Parameter space
Resolution in phase space
Resolution in parameter space
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and for all parameters

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**Input:** Nonlinear map, Phase space, Parameter space

Resolution in phase space
Resolution in parameter space
The Data Base

The Continuation Graph
The Data Base

The Continuation Graph

Nodes represent Conley–Morse Graphs
Conley-Morse Graphs

Saturday, July 3, 2010
The Data Base

Class 1
[890 boxes]

Class 2
[759 boxes]

Class 3
[251 boxes]

Class 4
[196 boxes]

Class 5
[88 boxes]

Class 6
[73 boxes]

Class 7
[66 boxes]

Class 8
[65 boxes]

Class 9
[50 boxes]

Class 10
[43 boxes]

Class 11
[12 boxes]

Class 12
[2 boxes]

Class 13
[1 box]

Class 14
[1 box]

Class 15
[1 box]

Class 16
[1 box]

Class 17
[1 box]

The Continuation Graph

Nodes represent Conley–Morse Graphs

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The Data Base

The Continuation Graph

Nodes represent Conley–Morse Graphs

Edges indicate connectivity in parameter space
The Continuation Diagram

Different colors represent different continuation classes.
Database results are never wrong, but they depend on the resolution!

Appropriate resolution is problem dependent!
Querying the Database: Are there multiple basins of attraction?
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Query the gradient-like structure:
Is there a Morse graph with multiple minimal elements?
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2 observable basins of attraction
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Querying the Database: **Are there multiple basins of attraction?**

**Query the gradient-like structure:**
Is there a Morse graph with multiple minimal elements?

2 observable basins of attraction

**Can we characterize the attracting dynamics?**
Querying the Database: Are there multiple basins of attraction?

Query the gradient-like structure: Is there a Morse graph with multiple minimal elements?

Can we characterize the attracting dynamics?

Query the Conley index:
Querying the Database: Are there multiple basins of attraction?

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Query the Conley index:

“3 cycle”  “1 cycle”
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"3 cycle" "1 cycle"
III. Theoretical Framework
What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$. 
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**Attractor block:** A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \text{int}(N)$$
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1. Measurement error
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**Remarks:**
1. The set of attractor blocks defines a (large) lattice under $\cap$ and $\cup$.
2. The separatrix dynamics is not explicit in the lattice of attractor blocks.
What about the dynamics?
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The **Omega limit set** \( \omega(N, f_{\lambda_0}) := \bigcap_{n=0}^\infty \text{cl} \left( \bigcup_{k=n}^\infty f_{\lambda_0}(N) \right) \)

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We can generalize this.

A compact set \( N \subset X \) is an **isolating neighborhood** for \( f_{\lambda_0} \) if the maximal invariant set in \( N \) lies in the interior of \( N \).

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\[ S = \text{Inv}(N, f_{\lambda_0}) \subset \text{int}(N) \]
A Morse covering of \( X \) consists of a finite poset \((P, \leq)\) that labels a collection of disjoint non-empty isolating neighborhoods \( B = \{ B(p) \mid p \in (P, \leq) \} \) with the property that given an orbit \( \gamma := \{ x_n \in X \mid n \in \mathbb{Z}, x_{n+1} = f(x_n) \} \) either

- there exists \( p \in P \) such that \( \gamma \subset B(p) \), or

- there exists \( q, p \in P \) and \( t_q, t_p \in \mathbb{Z} \) such that \( q < p \) and \( t_q > t_p \) for which

\[
\begin{align*}
\{ x_n \mid n \leq t_p \} & \subset B(p) \\
\{ x_n \mid n \geq t_q \} & \subset B(q) \\
\{ x_n \mid t_p < n < t_q \} \cap (B(p) \cup B(q)) & = \emptyset
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Prop: $M := \{ (p, M(p)) \mid p \in (P, \leq), M(p) = \text{Inv}(B(p)) \}$ is a Morse decomposition
A Discrete Representation of the Dynamics
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Choose a compact region in parameter space: $Q \subset \Lambda$
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Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \supseteq \mathcal{X}$
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Numerical/Experimental Error
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\[
f(G, Q) \subset \text{int}(\|F_Q(G)\|)
\]

Define a multivalued map: \( F_Q : \mathcal{X} \nrightarrow \mathcal{X} \)

Numerical/Experimental Error
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Choose a (cubical) grid \( \mathcal{X} \) that covers \( X \)

\[ f(G, Q) \subset \text{int}(|F_Q(G)|) \]

Define a multivalued map: \( F_Q: \mathcal{X} \rightrightarrows \mathcal{X} \)

Numerical/Experimental Error

\( F_Q \) is a directed graph:

Vertices \( G \in \mathcal{X} \)

Edges \( H \in F_Q(G) \Rightarrow G \rightarrow H \)
A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

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Numerical/Experimental Error: $\mathcal{F}_Q$ is a directed graph:

Vertices $G \in \mathcal{X}$

Edges $H \in \mathcal{F}_Q(G) \implies G \rightarrow H$

Recurrence in a Directed Graph

$\uparrow$ $\downarrow$

Strongly Connected Path Components
A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

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Recurrence in a Directed Graph

\[ \uparrow \]

Strongly Connected Path Components

1. Can be computed in linear time
2. Define a Morse Cover
Birkhoff’s Representation Theorem

\[ \text{O} \\downarrow \text{I} \]

\[ \text{O}(P) \]

\[ \text{J}^\vee \]

\[ \text{J}^\vee (\text{O}(P)) \]

\[ \mathbb{R} \]

\[ P \]
Birkhoff’s Representation Theorem

Finite Poset

\[ P \xrightarrow{O} O(P) \xrightarrow{J^\vee} J^\vee (O(P)) \xrightarrow{\mathbb{R}} P \]
Birkhoff’s Representation Theorem

Finite Poset

\[ P \rightarrow O \rightarrow O(P) \rightarrow J^\vee (O(P)) \rightarrow \mathbb{R} \rightarrow P \]

Posets

Category
Birkhoff’s Representation Theorem

Finite Poset

construct the collection of lower sets

\[ P \]

\[ O \]

\[ O(P) \]

\[ J^\lor \]

\[ J^\lor (O(P)) \]

\[ \mathcal{R} \]

\[ P \]
Birkhoff’s Representation Theorem

Category

Finite Poset

construct the collection of lower sets

Finite Distributive Lattice $\langle \cup, \cap \rangle$

Posets

$P$

$O$

$O(P)$

$J^\vee$

$J^\vee (O(P))$

$\subseteq$

$P$
Birkhoff’s Representation Theorem

Finite Poset

\( P \)

\( O \)

\( O(P) \)

\( J \lor \)

\( J^\lor (O(P)) \)

\( \mathcal{R} \)

\( P \)

Finite Distributive Lattice

\((\cup, \cap)\)

construct the collection of lower sets

Category

Posets

Lattices

contravariant functor
Birkhoff’s Representation Theorem

Finite Poset

construct the collection of lower sets

Finite Distributive Lattice \((u, \cap)\)

choose the join irreducible elements

Category

Posets

contravariant functor

Lattices
Birkhoff’s Representation Theorem

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Finite Poset

construct the collection of lower sets

Finite Distributive Lattice

(∪, ∩)

choose the join irreducible elements

Finite Poset

P

O

O(P)

J^∨

J^∨(O(P))

R

P

Posets

contravariant functor

Lattices

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Birkhoff’s Representation Theorem

Finite Poset

construct the collection of lower sets

Finite Distributive Lattice

(∪, ∩)

choose the join irreducible elements

Finite Poset

O(P)

P

O

P

Finite Poset

\( O(P) \)

\( J^\vee \)

Finite Poset

\( J^\vee (O(P)) \)

\( \mathcal{R} \)

\( P \)

Category

Posets

contravariant functor

Lattices

contravariant functor

Posets

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Birkhoff’s Representation Theorem

Finite Poset

construct the collection of lower sets

Finite Distributive Lattice \((\cup, \cap)\)

choose the join irreducible elements

Finite Poset

Birkhoff proved the existence of a poset isomorphism

\[ P \]

Category

Posets

contravariant functor

Lattices

contravariant functor

Posets

poset isomorphism

Posets

Lattices

Posets

Finite Distributive Lattice \((\cup, \cap)\)

\[ \text{O}(P) \]


\[ J^\vee (\text{O}(P)) \]
Birkhoff’s Representation Theorem

Category

Finite Poset
- construct the collection of lower sets

Finite Distributive Lattice \((\cup, \cap)\)
- choose the join irreducible elements

Finite Poset
- Birkhoff proved the existence of a poset isomorphism

Morse Decomposition
Combinatorial Theory

Morse Decomposition

M

O

O(M)

\( J^\vee \)

\( J^\vee (O(M)) \)

Birkhoff

M

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Combinatorial Theory

Morse Decomposition

M

O

O(M)

J^\vee

J^\vee (O(M))

Birkhoff

M

Structures of Nonlinear Dynamics

Nonempty Morse sets
In the Computer

Combinatorial Theory

Morse
Covering

B

Inv

Morse
Decomposition

O

O(M)

J\nu

J\nu (O(M))

Birkhoff

M

Structures of
Nonlinear Dynamics

Nonempty
Morse sets

pM

❄

id

❄

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M

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pM

O

Add Unstable Manifolds
O(\mu_M)

Finite Lattice of Attractors
A(pM)

Birkhoff

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Morse Covering

Structures of Nonlinear Dynamics

Combinatorial Theory

Morse Decomposition

Inv

Nonempty Morse sets

pM

O

O(M)\xrightarrow{O(\muM)} O(pM)

Add Unstable Manifolds

\nu

J\nu

A(pM)

Finite Lattice of Attractors

id_{pM}

J\nu \circ (O(M)) \xleftarrow{J\nu \circ (A(pM))} M

Birkhoff

Remove Orbits

\mu

pM

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Morse Covering

B

Morse Decomposition

O

O

M

O(pM)

O

Inv

Add Unstable Manifolds

\nu

Finite Lattice of Attractors

\nu

Remove Orbits

\mu

\nu_M

Nonempty Morse sets

pM

Structures of Nonlinear Dynamics

Combinatorial Theory
Structures of Nonlinear Dynamics

Combinatorial Theory

Nonempty Morse sets

Morse Decomposition

Conley Index provides a lower bound on image

Finite Lattice of Attractors

Add Unstable Manifolds

Move Forward by paths in $F$

Remove Paths

Remove Orbits

Birkhoff

Add Unstable Manifolds

Finite Lattice of Attractors

Morse sets

Nonempty Morse sets

Conley Index provides a lower bound on image

Morse Covering

In the Computer

Birkhoff

id

id

❄

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❄
Thank-you for your attention

http://chomp.rutgers.edu/

A Database Schema for the Analysis of Global Dynamics of Multiparameter Systems

SIADS, 8 (2009)

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