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A learning method for localizing objects in reverberant domains with limited measurements

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This article presents a learning (training)-based method for localizing objects in enclosures. Wave propagation in enclosures can lead to mixing of the wave energy, ultimately leading to incoherent spreading of information. This makes the localization problem challenging. However, spreading of the wave energy can lead to multiple interrogations of each point in the enclosure, which is in essence reminiscent of an ergodic or a closely ergodic behavior. Hence, any substructural changes in the enclosure can be sensed with sufficient information carried by the wave energy flow. Furthermore, temporal information buried in data makes it feasible to conduct only a few spatial measurements. A localization scheme is presented that benefits from the reverberant field and can reduce the required number of spatial measurements. © 2017 Acoustical Society of America.

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I. INTRODUCTION

Wave propagation in complex structures in the high frequency (small wavelength) limit is rich, complicated, and yet of practical use. The complexity of wave media can be either due to the presence of subwavelength inhomogeneities (scattering objects) of the wave velocity, or to the geometry boundaries enclosing a homogeneous medium. It is the latter case that is of interest in this study. Wave propagation in enclosures is of great practice in a broad range of applications such as room acoustics,1,2 ultrasonic monitoring of structural defects and damages,3,4 ultrasonic touchscreen,5–7 microwave navigation of objects in indoor environments,8 etc., all of which share in common an underlying physical nature, the reverberant field.

Reverberation is the process of formation of a wavefield in enclosures as a result of a large number of reflections.1 It leads to mixing of the wave energy, which in turn results in incoherent spreading of information. However, spreading of the wave energy can lead to multiple interrogations of each point in the enclosure, which is the reminiscent of an ergodic or a closely ergodic behavior.9,10 Hence, any substructural changes in the enclosure can be sensed with sufficient information carried by the wave energy flow.

Energy mixing and high frequency behavior of wave fields in bounded domains are reminiscent of the transient wave chaos. Wave chaos leads to rich wave phenomena such as universal statistical behaviors of the frequency spectra and certain spatial patterns of the modes of the corresponding enclosure.9,11,12 The statistical behavior of the high frequency modes are dependent upon the geometry of the enclosure. In a classical setting, there are two types of motion: regular (integrable) and irregular (chaotic). Regular domains have stable trajectories and may also exhibit caustics, i.e., regions that the ray trajectories never visit regardless of

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the number of reflections. In chaotic domains, on the other hand, the trajectories are unstable and ergodic, meaning they interrogate all the points in the wave domain almost surely. Instability of the ray trajectories in wave enclosures is the manifestation of the extreme sensitivity to the inputs of the system. Furthermore, the geometry of regular systems can be sensitive to any perturbation to the geometry so that any irregular perturbation in the order of a wavelength can turn it into a chaotic domain.11 High frequency eigen modes of a bounded wave domain can be speckle-like (ergodic). An ergodic typical mode can be viewed as a Gaussian random function, resulting from a random superposition of plane waves.11,13 An ergodic behavior motivates that information will reach out everywhere with equal probability. When an object in placed in an otherwise homogeneous regular domain, it effectively perturbs the base wave properties such as the refractive index; this in turn perturbs some of the base modes by ergodic ones.

Incoherent propagation and information mixing in enclosures make the localization problem very challenging. In classical imaging techniques, a set of measurements are collected by a selective combination of sources and receivers of the wave energy (e.g., transducers), packed together closely (in the order of a wavelength). The methods heavily rely on the first arrivals of the perturbations and also the compactness of the collected information in time, where the primaries and reflections can be distinct, and higher orders of information are progressively of less practical use. In a reverberant field, one the other hand, the information of the field-perturbing object(s) can be well-mixed. Reflections may not be distinct and can even potentially contaminate the first arrival; and in turn, this would degrade the performance of the imaging algorithm. Furthermore, reverberant fields are generally diffusive in a long timescale and classical inversion models based on the regular perturbation theory such as the transient Born approximation break down in a long timescale.
Classical imaging techniques also require the data to be registered at a large number of spatial points. Spatial measurements, however, in some applications can be costly in several different ways including acquisition time, computational processing, and equipment. Exploring the reverberant data, however, gives the feasibility of reducing the number of spatial measurements. One of the key assumptions in classical imaging is that the domain being imaged is semi-infinite and non-compact, and the radiation condition is indeed in effect, implying there is no physical boundaries off which the waves can reflect. Even in bounded domains the transducers are arranged such that the higher order reflections can be muted. This is a base theme of many imaging technologies such as ultrasound, radar, and seismic imaging techniques. Naive applications of free-space Born imaging methods to a reverberant field produce meaningless results.

Training or learning methods have been previously applied in different contexts such as localization and classification of defects and flaws in solid substrates, localization of tactile objects in contact with a plate, and localization by probabilistic pattern recognition both in reverberant and non-reverberant fields. Generally, the system is looked at as a black box, and any (a posteriori) measured data are matched with a set of a priori measured data. We present a method to localize finite objects (i.e., objects with characteristic dimensions in the order of a wavelength) in a highly reverberant field. The objects are assumed to be compactly distributed in a bounded domain. We consider the localization of passive objects; that is, the waves are generated and received by a set of transducers, and the objects perturb the wave field upon interfacing with the wave energy flow.

The organization of the paper is as follows. In Sec. II, we present a canonical model that governs the physics of the imaging system, which can be universal in interpretation. We next pose the localization problem as an inverse problem in Sec. III. We then present the theory and derivation of the proposed method in Sec. IV. Following this, we present possible variations for further improvements in Sec. V. Further properties of the method are discussed in Sec. VI. Robustness analysis in presented in Sec. VII. The concepts are then verified through a set of numerical tests in Sec. VIII and a summery concludes the paper.

II. MODEL PROBLEM

Physics underlying imaging problems are usually governed by wave propagation or its variants. They are represented by a set of partial differential equations with real characteristic curves (i.e., space-time curves along which information propagates). Let \( \partial_t \) and \( \partial_x \), respectively, indicate the first and second partial derivatives in time. The most general linear form of this type can be written as

\[
\begin{align*}
\{A(m)\partial_t + B(n)\partial_x + K\} \Phi &= F, \\
\end{align*}
\]

where \( m \) and \( n \) are spatial functions that can parametrize the propagation, such as a combination of speed of sound, density, absorption coefficients, etc., \( \Phi \) is the wave field such as the acoustic pressure field in rooms or indoor environments or the displacement field in solid substrates, \( F \) is the source field by an acoustic source, ultrasonic transducer, or antenna. \( A \) and \( B \) are generally considered as linear operators of \( m \) and \( n \), \( K \) is generally a uniformly elliptic operator that is symmetric positive definite. Two canonical examples are the wave and plate equations or any other form that can be derived from a system of hyperbolic equations. For practical purposes, \( K \) is discretized using a numerical scheme such as finite differences or finite elements, whereby \( K \) becomes a symmetric positive definite matrix. The boundary conditions are effectively encapsulated inside the discrete representations of \( K \) and \( F \).

In practice, the wave field \( \Phi \) is not measured at every point in space, but only at a subset of points. This process can be identified by defining a down sampling operator \( \mathcal{P} \), i.e., a matrix whose elements are either 1 if the corresponding point is being observed or 0 otherwise.

In the present paper, the methods are evaluated through numerical simulations of a wave equation of the following type:

\[
\left( \frac{\rho(x)}{c_0^2} \partial_t + \sigma(x) \partial_x - \Delta \right) q(t) \delta_{h}(x), \quad \text{in } \Omega,
\]

\[
\phi = 0, \quad \text{on } \partial \Omega,
\]

\[
\phi(\cdot, t = 0) = \partial_t \phi(\cdot, t = 0) = 0,
\]

where \( \Omega \) represents the wave domain and \( c_0 \) is the speed of propagation with some spatial perturbations represented by \( \rho, \sigma \) is the effective absorption. Upon discretization, it gives the following representation:

\[
\{A(m)\partial_t + B(n)\partial_x + K\} \Phi = F,
\]

\[
A(m) = \frac{1}{c_0^2} m, \quad B(n) = n I.
\]

Nonetheless, the methods will be presented and derived based on Eq. (1), which represents the localization problem in wave enclosures in its full generality and the operators \( A, B, \) and \( K \) can be adjusted to the context of interest. Moreover, it will be shown later that the final algorithms are independent of these operators.

III. LOCALIZATION AS AN INVERSE PROBLEM

Localization of an object can be formulated as an inverse problem. Let \( \mathcal{L} \) be the (nonlinear) operator that maps the parameter functions to the measured data \( d \). Let \( \mathcal{D} \) be the data space spanned by the measurements corresponding to each source-receiver pair. The localization problem can be expressed then as

\[
(m, n) = \arg \min_{(m, n)} \frac{1}{2} \left\| \mathcal{L}(m, n) - d \right\|_{L^2(D \times [0, T])}^2.
\]

This poses the challenge of estimating the operator \( \mathcal{L} \), which is inherently nonlinear. Among literature, the most common techniques include the Continuum finite differences or finite elements, and boundary integral methods. Former is
predominant in seismic imaging,\textsuperscript{15} where a real-time implementation is not crucial. The latter is in practice in radar and ultrasound\textsuperscript{14,16} imaging, where only imaging primary echoes is of concern. This representation can be used to reformulate the inversion as
\[
\min_{(m,n)} \frac{1}{2} \| \mathbf{P} \Phi - \mathbf{d} \|^2_{L^2([D \times [0,T])},
\]
(5a)
subject to
\[
\{ A(m)\partial_x + B(n)\partial_t + K \} \Phi = F.
\]
(5b)

Generally, in order to solve such an optimization problem, the gradient of the objective functional with respect to the parameter functions is required. One possible approach is to construct a set of grid points for the parameter functions. For each grid point, the constraint is solved in order to calculate a corresponding field value, and hence, the corresponding value of the objective functional. The functional gradient can next be approximated by means of a finite differencing scheme. This approach can be potentially expensive, in practice. The learning theory presented here provides an alternative and more efficient way of addressing this method.

\section*{IV. LEARNING APPROACH}

Consider the model problem, Eq. (3), in which we are interested to reconstruct the parameters \( \mathbf{m} \) and \( \mathbf{n} \). The relation between the parameters \( \mathbf{m} \) and \( \mathbf{n} \) and the field function \( \Phi \) is nonlinear. Consider a base state \( \Phi_o \), which corresponds to a field with no perturbations in the parameters \( \mathbf{m} \) and \( \mathbf{n} \); that is, the base state satisfies
\[
\{ A(m_o)\partial_x + B(n_o)\partial_t + K \} \Phi_o = F.
\]
(6)

Imposing the perturbations \( \delta \mathbf{m} \) and \( \delta \mathbf{n} \) over the parameters \( \mathbf{m}_o \) and \( \mathbf{n}_o \) will consequently induce a perturbation \( \delta \Phi \) over the wave field \( \Phi_o \). Consider \( \Phi = \Phi_o + \delta \Phi \), \( \mathbf{m} = \mathbf{m}_o + \delta \mathbf{m} \), \( \mathbf{n} = \mathbf{n}_o + \delta \mathbf{n} \), and replace these expressions in the model problem to obtain
\[
\{ A(m_o + \delta m)\partial_x + B(n_o + \delta n)\partial_t + K \} (\Phi + \delta \Phi) = F.
\]
(7)

For simplicity, it is useful to introduce the following definitions:
\[
\mathbf{L} := \mathbf{L}_o + \delta \mathbf{L},
\]
(8a)
\[
\mathbf{L}_o := \{ A(m_o)\partial_x + B(n_o)\partial_t + K \},
\]
(8b)
\[
\delta \mathbf{L} := \{ A(\delta m)\partial_x + B(\delta n)\partial_t \}.
\]
(8c)

We can then write
\[
\mathbf{L}\delta \Phi = -\delta \mathbf{L}\Phi_o.
\]
(9)

Note also that \( \delta \mathbf{L} \) can be rearranged as
\[
\delta \mathbf{L} = \delta \Sigma \left\{ \frac{\partial_o}{\partial_t} \right\}, \quad \delta \Sigma = \left\{ \begin{array}{c}
A(\delta m) \\
B(\delta n)
\end{array} \right\}.
\]

The intuitive idea behind the learning approach emanates from projecting the function \( \Sigma \) over a finite basis set of simple functions; that is, suppose
\[
\delta \Sigma(x) \approx \sum_{i=1}^{N} \theta_i \chi_{n_i}(x, a_i), \quad \theta_i \in \mathbb{R},
\]
(10)

where \( \chi_n(x, a_i) \) is an indicator function centered at \( x_i \) with \( |\text{supp}(\chi_n(x, a_i))| \approx a_i^2 \); in other words, the induced perturbations by a collection of objects can be constructed by summing over some reference objects.

Next consider an orthogonal Euclidean basis \( \{ \Phi_i \}_i \) for \( \mathbb{R}^N \). Let \( \Phi_i \) be the field corresponding to the parameter function \( \delta \Sigma(x) = \chi_n(x, a_i) \). In other words,
\[
\mathbf{L}_o^\dagger \delta \Phi_i = -\delta \mathbf{L}_o \Phi_o.
\]
(11)

Note that for each basis in the parameter space, one could have a similar equation. Let \( \delta \Sigma = \sum_{i=1}^{N} \theta_i \delta \Sigma_i \), and thus, \( \delta \mathbf{L} = \sum_{i=1}^{N} \theta_i \delta \mathbf{L}_i \). Next, by multiplying each equation by \( \theta_i \), and summing over \( i \), one would get
\[
\sum_{i=1}^{N} \theta_i \mathbf{L}_o \delta \Phi_i = -\sum_{i=1}^{N} \theta_i \delta \mathbf{L}_o \Phi_o = -\delta \mathbf{L} \Phi_o.
\]
(12)

Observe that based on the orthogonality of \( \theta_i \)'s, we can write
\[
\sum_{i=1}^{N} \theta_i \delta \mathbf{L}_o \delta \Phi_i = \sum_{i=1}^{N} \theta_i \delta \mathbf{L}_i \delta \Phi_i
\]
\[
= \sum_{i=1}^{N} (\theta_i^2 - \theta_i) \delta \mathbf{L}_o \delta \Phi_i + \sum_{i,j=1, i \neq j}^{N} \theta_i \theta_j \delta \mathbf{L}_i \delta \Phi_j
\]
\[
= \sum_{i=1}^{N} (\theta_i^2 - \theta_i) \delta \mathbf{L}_o \delta \Phi_i.
\]
(13)

Thus, upon the orthogonality and conditions \( |\theta_i| \approx 0 \) or \( |\theta_i - 1| \approx 0 \), for all \( 0 \leq i \leq N \), we have
\[
\sum_{i=1}^{N} \theta_i \theta_j \delta \mathbf{L}_o \delta \Phi_i \approx \left( \sum_{i=1}^{N} \theta_i \delta \mathbf{L}_i \right) \left( \sum_{j=1}^{N} \theta_j \delta \Phi_j \right)
\]
\[
= \delta \mathbf{L} \left( \sum_{j=1}^{N} \theta_j \delta \Phi_j \right).
\]
(14)

Also, it can be easily seen that
\[
\sum_{i=1}^{N} \theta_i \mathbf{L}_o \delta \Phi_i = \mathbf{L}_o \left( \sum_{i=1}^{N} \theta_i \delta \Phi_i \right).
\]
(15)

Let \( \Phi = \sum_{i=1}^{N} \theta_i \delta \Phi_i \), which in turn gives
\[
\mathbf{L}_o \delta \Phi_o = -\delta \mathbf{L} \Phi_o.
\]
(16)

Consider a case where the parameter \( \delta \Sigma \) is exactly expressible in terms of the basis functions, i.e., Eq. (10) is exact, resulting in \( \delta \Sigma = \delta \Sigma \), and \( \theta_i \in \{0, 1\} \), for all
0 \leq i \leq N$. Then by the uniqueness of the solution of the model problem, it can be concluded that

$$\delta \Phi = \tilde{\delta} \Phi = \sum_{i=1}^{\infty} \theta_i \delta \Phi_i,$$

(17)

Namely, the solution corresponding to an arbitrary set of parameter functions is a linear combination of the measurements corresponding to the prior parameter functions and with the same projection coefficients. In most cases, however, there could be some error in Eq. (10). Nonetheless, Eq. (17) can be taken as a good approximation to the exact perturbed field. The resolution of such an approximation obviously depends upon how well the parameter functions can be approximated by the assumed set of simple functions.

Next, upon down sampling the field at the measurement points, we get

$$\delta d \approx \sum_{i=1}^{N_r} \theta_i \delta d_i, \quad \delta d_i = \mathcal{P} \delta \Phi_i,$$

(18)

where $\delta d_i$ is the system response to the parameter function $\chi_i(x, a_i)$.

We remark that in the derivation above no linearization was assumed (unlike conventional Born methods). Note also that this method merely requires measurements (observations); thus, it calls for a limited knowledge about the system. This approach, upon utilizing the entire reverberant field and long time data, requires a very limited number of spatial measurements (one or two). Furthermore, since the linear combination is pointwise in time, the entire representation is independent of aliasing in time. Hence, it offers a great flexibility in terms of the sampling requirements in space or time.

A. Mathematical interpretation

Note that Eq. (18) is independent of the propagation operators $L$ and $\delta L$. This implies the system can now be looked at as a black box, for which a limited knowledge may be available. This offers an experimental approach to this problem; in cases that computing the operator $L$ is difficult, the learning theory can be utilized to teach an alternative operator $\mathcal{M}$ by training the system by a prior set of reference measurements, which will be henceforth referred to as the training set.

Mathematically, this method is reminiscent of considering the references as bases for a vector space spanned by the training set and then trying to find the projection of an arbitrary measurement in that space. The operator $\mathcal{M}$ can be thought of as a matrix with $N$ columns and infinite rows (experimentally very large, $\approx 10^2$), i.e., a matrix with the reference measurements as the columns. We remark also that the reference measurements may not generally be orthogonal (for a weak object, in fact, they can be very close). Let $\delta d(t)$ be a measurement, and $\tilde{\delta} d(t) = \tilde{\delta} d^{(t)}(t) + \tilde{\delta} d^{-(t)}(t)$, the orthogonal decomposition of it, where $\tilde{\delta} d^{(t)}(t) \in \mathcal{D}$, $\tilde{\delta} d^{-(t)}(t) \in L^2 \setminus \mathcal{D}$, and $\mathcal{D} = \text{span}\{\delta d_i(t)\}_{i=1}^{N}$. The projection operator in terms of the data matrix is $\mathcal{M}(\mathcal{M}^t \mathcal{M})^{-1} \mathcal{M}^t$, where $\mathcal{M}^t$ is the adjoint of $\mathcal{M}$. Hence,

$$\tilde{\delta} d = \mathcal{M}(\mathcal{M}^t \mathcal{M})^{-1} \mathcal{M}^t \delta d.$$

(19)

Upon projecting an arbitrary measurement onto the training data space, we expand it as a linear combination of the bases (i.e., the reference measurements). That is to write $\delta d^{(t)}(t) = \sum_{i=1}^{N} \theta_i \delta d_i(t) = \mathcal{M} \Theta$, $\Theta \in \mathbb{R}^N$, $\Theta = \{\theta_1, \ldots, \theta_N\}^t$. Combining this with the previous equation gives

$$\Theta = (\mathcal{M}^t \mathcal{M})^{-1} \mathcal{M}^t \delta d,$$

(20)

which is equivalent to solving a least squares problem:

$$\min_{\Theta \in \mathbb{R}^N} \frac{1}{2} \| \mathcal{M} \Theta - \tilde{\delta} d \|^2_{L^2([0,T])},$$

(21)

When there exist a number of sources and receivers (say $N_s$ and $N_r$, respectively), we can extend the formulation above to

$$\min_{\Theta \in \mathbb{R}^N} \frac{1}{2} \sum_{r,s} \mu_{r,s} \| \mathcal{M}_{r,s} \Theta - \delta d_{r,s} \|^2_{L^2([0,T])},$$

(22)

where $\mu_{r,s}$ is the weighting parameters, $\mathcal{M}_{r,s}$ and $\delta d_{r,s}$ are the data matrix and the measured signal at the $r$th receiver in response to the $s$th source.

V. VARIATIONS OVER THE BASIC ALGORITHM

As aforementioned, the measurements can be close to one another (in the energy norm) in the training data space. Furthermore, measuring or constructing the perturbation $\delta d$ generally is not a robust approach, since it should be subtracted from $d_r$ (the measurement with no field-perturbing objects present in the medium). A more robust alternative is to augment the data space by the base measurement $d_r$, with the corresponding projection coefficient $\theta_r$, in which case it results in a new constraint:

$$\sum_{i=1}^{N} \theta_i = 1.$$

(23)

The underlying physics motivates to enforce a positivity constraint. This is because of the positive definiteness and stability of the system, for which we require $m(x), n(x) \geq 0$.\textsuperscript{20} The physical interpretation is that the entire system (including the object and medium) either conserves or loses the total wave energy. This, in turn, leads to a positivity constraint: $\theta_i \geq 0$, for all $i$. Furthermore, when it is believed that the distribution of objects is sparse, the optimization problem can be penalized by a sparsity promoting constraint. This is practically imposed by penalizing the problem through the $l_1$ norm.\textsuperscript{23,24} However, given the constructed constraints, the overall scheme can be conveniently implemented as

$$\min_{\Theta \in \mathbb{R}^N} \frac{1}{2} \sum_{r,s} \mu_{r,s} \| \mathcal{M}_{r,s} \Theta - d_{r,s} \|^2_{L^2([0,T])},$$

(24a)
subject to
\[ \theta_i \geq 0, \text{ for all } i, \] (24b)
\[ \mu \sum_{i=1}^{N} \theta_i = 1. \] (24c)

\( \mu \) is a penalty parameter. This variant can improve the success of the localization. However, another more powerful variant can be introduced by reformulating the problem in the image space as opposed to the data space. The space spanned by all possible configurations of \( \Theta \) is called the image space, denoted by \( \mathcal{I} \). This suggests posing the localization problem as a minimization in the image space with essentially the same constraints as before. That is,

\[
\min_{\Theta \in \mathbb{R}^N} \frac{1}{2} \sum_{r,s} \mu_{r,s} \| \Theta - (\mathcal{M}_{r,s})^{-1} \mathcal{M}_{r,s} d_r \|_{L^2(\mathcal{I})},
\]

subject to
\[ \theta_i \geq 0, \text{ for all } i, \]
\[ \mu \sum_{i=1}^{N} \theta_i = 1. \]

This algorithm can be implemented as a two-step method: Step (1)—solve the original unconstrained least squares,

\[
\Theta^*_{r,s} = \arg \min_{\Theta_{r,s} \in \mathbb{R}^N} \frac{1}{2} \| \mathcal{M}_{r,s} \Theta_{r,s} - d_r \|_{L^2(\mathcal{I})},
\]

Step (2)—solve a constrained least squares as follows:

\[
\min_{\Theta \in \mathbb{R}^N} \frac{1}{2} \sum_{r,s} \mu_{r,s} \| \Theta - \Theta^*_{r,s} \|_{L^2(\mathcal{I})},
\]

subject to
\[ \theta_i \geq 0, \text{ for all } i, \]
\[ \mu \sum_{i=1}^{N} \theta_i = \sum_{i=1}^{N} \theta_i, \]

VI. REMARKS ON OPERATING ON THE PROJECTION PROBLEM

In general, one could multiply each side of Eq. (18) by any matrix, e.g., a time-down sampling operator. This, however, may not always lead to a working method. The necessary condition for the operation to work is the invertibility of the operator acting on both sides. If the operator has a finite null space, it effectively maps the data onto a lower dimensional space, which in turn may lead to loss of information. Invertibility guarantees that the base algorithm (i.e., with no penalization or constraint) provides an identical result to the untransformed equation. The sufficient condition depends on the imposed constraints, which may produce a better or worse performances; both are likely.

An immediate and equivalent formulation is the projection problem in the frequency domain, i.e.,

\[ \sum_{i=1}^{N} \theta_i \delta \hat{d}_i(\omega) = \delta \hat{d}(\omega), \text{ for all } \omega \in [\omega_{\text{min}}, \omega_{\text{max}}], \]

where \( \hat{d}(\omega) \) indicates the Fourier representation of \( d(t) \), and \([\omega_{\text{min}}, \omega_{\text{max}}]\) represents the bandwidth of the data. In practical applications, the wave transducers are generally band-limited. Hence, only a subset of frequencies are needed to essentially capture the observable information. This consequently leads to a significant reduction of the number of equations that need to be solved, depending on the bandwidth, sampling frequency, and timespan of data. Utilizing a fraction of information over the Fourier bases, namely, the amplitudes or just the real parts, are also among nonlinear operations that can produce a working algorithm.

Other formulations given by nonlinear operations with some potential benefits in some applications are

\[ \sum_{i=1}^{N} \theta_i |\delta d_i(t)| = |\delta d(t)|, \text{ for all } t \in [0, T], \]

\[ \sum_{i=1}^{N} \theta_i |(\mathcal{H} \delta d_i)(t)| = |(\mathcal{H} \delta d)(t)|, \text{ for all } t \in [0, T], \]

where \( |\delta d(t)| \) is the modulus of \( \delta d \) at time \( t \) and \( \mathcal{H} \) is the Hilbert transform. Note that \( |(\mathcal{H} \delta d)(t)| \) essentially gives the envelope of \( \delta d \).

A. Sparse sampling in space and time

Note that the linear combination 18 is point-wise in time and space. Down sampling in space reflects the fact that this method can be implemented using only a few measurements in space. In time, we take a linear combination of the prior knowledge to predict the measured data at the same instant in time. This provides a room for aliasing, provided that all measurements are aliased in the same way. This is more of a practical strategy to reduce the number of equations required to be solved in the localization process.

B. Interpolation from a Coarser to a Finer Training Set

A natural question is whether or not a finer training set (i.e., finer separation between the prior parameter function) can be estimated using a coarser one, i.e., estimating the response to an object at \( x \) based on the responses to the objects at the points \( x + z \Delta x, \ z \in \mathbb{R}^d \), in its neighborhood. A naive implementation is to take the linear combination of the adjacent points in time. This would fail, however, as this operation violates the independence of the bases. Nevertheless, a remedy can be proposed in cases where the relation between the perturbations in the parameter function and the field is linear (e.g., the Born approximation) or in cases that one can simply assume this linearity. Consider the training operator \( \mathcal{M} \) and the Born operator \( \mathcal{M} \). It is easy to see that \( \mathcal{M} = \mathcal{M} \mathcal{X} \), where \( \mathcal{X} \) maps the experimental training functions to the high resolution numerical ones (e.g., the
finite difference one). In order to change the resolution numerically, one should first calculate the pseudoinverses \( Y \) and \( Z \) of \( X \) such that \( XY = I \) and \( ZX = I \). Consider \( X', Y', Z' \), and \( M' \) corresponding to the new (finer) resolution. Then, one could write

\[
M = MY = M'Y'.
\]

Hence,

\[
M' = MYZ'.
\]

The pseudoinverse relations are given as

\[
Y = X^T(XX^T)^{-1}, \quad Z = (X^T)^{-1}X^T.
\]

C. Probabilistic interpretation

Suppose we postulate that the probability distribution of the registered data, \( \delta d_i \), at an instant in time given the prior measurements is a Gaussian parametrized by \( \theta_i \)'s with a finite variance \( \sigma \); and, that all \( \theta_i \)'s are equally likely, i.e.,

\[
p(\delta d_i|\Theta) \sim \exp \left(-\frac{(\delta d_i - \Theta_i \delta d_i)^2}{2\sigma^2}\right),
\]

and assuming the samples \( \delta d_i, \ldots, \delta d_n \) are

1. iid (independent identically distributed) random variables; that is,

\[
p(\delta d_i|\delta d_n; \Theta) = p(\delta d_i|\delta d_i; \Theta), \quad \text{for all } i, j, k, l.
\]

2. Markovian; that is,

\[
p(\delta d_i, \delta d_j, \ldots, \delta d_n|\delta d_n; \Theta) = p(\delta d_i|\delta d_{i-1}; \Theta),
\]

we can then estimate the joint probability distribution as

\[
p(\delta d|\Theta) = \prod_{i=1}^N p(\delta d_i|\Theta).
\]

Note that in this context, the time data are the samples (or observations) and the projection coefficients are the predictors (or estimators). The maximum likelihood estimate states that the best predictor is the one that maximizes the log-likelihood function; \( \hat{\Theta} = \arg \max_\Theta \hat{\ell}(\Theta|\delta d) \),

\[
\hat{\ell}(\Theta|\delta d) = \log p(\delta d|\Theta),
\]

which leads to

\[
\min_{\Theta \in \mathbb{R}^n} \frac{1}{2} ||M(\Theta - \delta d)||^2_{L^2([0,T])},
\]

which is identical to the projection formulation (21). This suggests that the underlying probability law governing the error in measuring the wave field is Gaussian. This observation is useful for further improvements by studying the cross-correlations of the training data, and augmenting the projection algorithm in several different ways such as introducing a well-estimated covariance matrix, which in turn would result in a weighted least squares formulation.

VII. ROBUSTNESS

Imaging systems with bounded domains have a finite bandwidth, which is reminiscent of the quality factor of the system itself or due to the transducers. Whence, only a limited bandwidth of the information is registered. There could generally be two types of noise sources. (1) Additive noise, which appears as high frequency fluctuations with generally a normal probability distribution. This noise can be easily filtered by a basic infinite impulse response or finite impulse response filter. (2) Multiplicative noise, which can be viewed as a convolution of a random function with the underlying true response of the system. Filtering this type of noise (which will be henceforth referred to as drift) can be challenging. It is, to a large extent, unknown and uncertain, and cannot be estimated or controlled to the precision required for the inversion process. Many factors can potentially contribute to this noise type, such as temperature, temperature gradient, mechanical noise and uncertainties coming about due to stresses and fatigue in time. This motivates to construct a methodology for a blind estimation and compensation of the drift, which can be applied to adapt a posterior training set to the prior one. Let \( \{\delta d_{i,j}\}_{j=0}^N \) and \( \{\delta d_{i,j}\}_{j=0}^N \) be, respectively, the posterior (drift affected) and prior (drift free) training sets. This process may go by different names such as restoration, registration, or deblurring.

Suppose \( D_r : L^2([0,T]) \rightarrow L^2([0,T]) \) is the operator that maps the posterior base (background) measurement (i.e., with no objects or perturbations) to the corresponding prior one, i.e.,

\[
d_r(t) = (D_r \delta d_i)(t).
\]

In the limit that drift-induced perturbations are small, one could consider \( D_r \) as a linear operator. We also require \( D_r \) to be time-shift invariant, which in turn concludes it should be of a convolution type operator. This reduces the problem to a blind deconvolution problem. Furthermore, we postulate that the drift source affects all the training measurements uniformly; that is, \( D_r \) is independent of \( i \), the measurement index. This makes it feasible to construct the drift operator using the known prior and posterior background states. In other words, \( D_r \) is estimated using \( \delta d_i \) and \( \delta d_o \), and then applied to map any measurement with unknown objects in the medium to the corresponding undrifting one, which can in turn be matched with the prior training set.

A. Drift operator as a Wiener filter

The goal is to estimate an operator \( D_r \) such that

\[
d_r(t) = (D_r \delta d_i)(t).
\]

Taking the Fourier transform of both sides yields \( e_r(\omega) = D_r(\omega) e_o(\omega) \), which gives \( D_r(\omega) = e_r(\omega)/e_o(\omega) \). This
can be rewritten as \( D_r(\omega) = \frac{[e_r/e_o]}{[e_r/e_o] + \epsilon} \), where we can introduce a regularization parameter \( \epsilon \) to account for the deviation around the null frequencies (they may exist both outside and inside the bandwidth of the system)

\[
D_r(\omega) = \frac{e_o}{e_o + \epsilon}.
\]

(39)

This is an example of a regularized inverse filtering. Now we can also add an additive noise to the system, which can be thought of as the difference between the white noise in the prior and posterior data models

\[
d_r(t) = (D_r \tilde{d}_r)(t) + n(t).
\]

(40)

A Wiener filter attempts to construct \( D_r \) such that the expected value of the energy of the error \( n(t) \) is minimized,

\[
D_r = \text{arg min}_{D_r} E[|n|^2].
\]

(41)

This gives

\[
D_r(\omega) = \frac{e_o}{e_o + \epsilon} S_{dd},
\]

(42)

where, \( S_{dd}, S_{nm} \) are the (auto)power spectral densities of the measurement and noise and \( D_r(\omega) \) is the Fourier kernel of \( D_r \). It can be shown that the Wiener filter is optimal when \( \epsilon = S_{nm}/S_{dd} \).

In practice, upon registering the background field, the drift operator is constructed as shown above. Next, an arbitrary measurement that corresponds to an unknown object is mapped to the corresponding prior model using the drift operator,

\[
d(t) \approx (D_r \tilde{d})(t),
\]

(43)

d(t) can now be used with the prior training set.

VIII. NUMERICAL TESTS

A. Simulation of the forward problem

In order to verify the proposed methods, the model problem, Eq. (3), was discretized using a fourth-order finite difference scheme in space with a second order two-step time integration. In the model, we set \( c_0 = 4000 \text{ m/s}, m_s = 1, \text{ and } n_o = 0.0001 \). The reverberant domain is considered as a \( 120 \text{ mm} \times 80 \text{ mm} \) two-dimensional rectangular domain. The source and receiver are placed at the short edges. The source is located at 20 mm from the long edge and the receiver is located at the opposite edge with a 40 mm offset from the source. The input source profile is a Gaussian-modulated sine wave with a center frequency at 900 kHz with a 35% bandwidth. Ten points per minimum wavelength with CFL = 0.2 were used to achieve a satisfactory accuracy and avoid dispersion errors. The field-perturbing objects have a circular intersection with a radius of 2.5 or 5 mm (in the order of a wavelength) and with around 1% perturbation over the base parameters. The time-span of the simulated data is 1 msec. The localization algorithms were solved using the least squares or quadratic programming solvers of the MATLAB R2014a (The Mathworks Inc., Natick, MA) optimization toolbox. In the followings, different sets of simulation results are presented in order to assess the performance of the localization algorithms and identify the features and limitations.

B. Localization of objects in a noise free field

For cases of one [Figs. 2(a) and 3(a)] and three randomly distributed [Fig. 4(a)] test objects, we performed a set of simulations to evaluate the performance of the projection method, Eq. (21), and the image space method, Eq. (25). In the second case, the objects have different sizes. In simulating the synthetic data, we set \( \delta m_o = 0.01 \) and \( \delta n_o = 0.00001 \). The domain was segmented into a \( 20 \times 12 \) rectangular grid with equal spacings, over which it was trained. An example of a training unit is shown in Fig. 1. The center of the training unit is placed sequentially at different locations over each node of the rectangular grid, and in response to each of the said configurations, a corresponding training (reference) measurement was simulated. The training measurements were then stored as the training set. Note that the test objects can be approximately considered as the superposition of the training units. The localization results using the projection algorithm (21) and the image space algorithm (25) are shown in Figs. 2, 3, and 4, where the locations and supports of the objects can be identified with sufficient contrast with respect to the surrounding domain. Furthermore, as evident, the image space method provides a significant improvement of the contrast of the objects.

C. Localization of objects with additive and drift noises

For a case of five objects, identical in size and randomly distributed, we performed a set of simulations with additive white noise (i.e., Gaussian independent identically
FIG. 2. (Color online) Localization of a single object using a 20 × 12 training set, using one source-receiver pair. (a) Exact object configuration. (b) Projection method. (c) Image space method.

FIG. 3. (Color online) Localization of a single object using a 20 × 12 training set, using one source-receiver pair. (a) Exact objects configuration. (b) Projection method. (c) Image space method.

FIG. 4. (Color online) Localization of three randomly distributed objects with different sizes using a 20 × 12 training set, using one source-receiver pair. (a) Exact objects configuration. (b) Projection method. (c) Image space method.
distributed random time samples) leading to 24 dB signal-to-noise-ratio of the recorded waveforms, using the same parameter inputs used in the previous tests. The localization results corresponding to progressively adding more sources and receivers are shown in Fig. 6, using the projection method. We remark that the white noise was added to the registered signal at the receiver. The domain was trained using the same circular objects, where the domain was subdivided into $16 \times 8$ training regions. As expected, adding more sources and receivers have improved the localization results.

In the second set of tests, we evaluated the effect of the drift noise using the same training set of the previous example. To do so, in simulating the synthetic measurement and environmental noise, we perturbed the speed of sound up to 0.001% as the first scenario, aiming to induce a gentle drift noise. Perturbations or changes of speed of sound or other material properties such as density, Young’s modulus, refractive index, etc. are reminiscent of the presence of environmental noise such as thermal fluctuations, humidity, etc. The attempts to improve the result with presence of the drift, through adding more sources and receivers evenly distributed at the boundaries, are represented in Fig. 7, suggesting that this strategy is not effective; and, this is the premise of applying the inverse filtering technique, Eq. (43). In Fig. 8, the performance of the drift compensating algorithm is presented using only one source and one receiver. This time the speed of sound was perturbed up to 2.5%. With no drift present, the result of the localization is shown in Fig. 8(a), corresponding to the ideal scenario. With drift present but no compensation, the result of the localization is shown in Fig. 8(b), where no object can be identified. Figure 8(c) demonstrates the successful recovery of the objects once the drift-affected measurement was compensated for the drift.

Note that in the simulations it was assumed that the sources and receivers have infinite bandwidths. In practice, however, they will have finite bandwidths, which are likely affected by the noise. However, the method presented here is of a blind type and attempts to calibrate the entire frequency response of the system, including the enclosure and transducers together. Therefore, it should work equally well on systems with finite-bandwidth sources and receivers.

D. Resolution analysis

A natural question arising in the context of any imaging technique is about the resolution limit, identified by the minimum size of a resolvable (identifiable) object. It is essentially diffraction limited and generally in the order a half-wavelength in classical techniques. The answer to this question, however, is more subtle for the proposed learning algorithm, as the entire reverberant field is treated. In order to quantify an answer, a series of simulations were performed, where a circular object [Fig. 9(a)] is placed at two distinct locations several wavelengths apart ($\approx 36$ mm) over a $120 \times 80$ mm reverberant domain with the source and receiver at the same locations as in the first example. In the series of simulations, the radius of the object is scaled with respect to the wavelength at each location separately and the strength of the corresponding perturbed field at the receiver is plotted in Fig. 9(b) according to

$$e = \frac{\|d - d_o\|_{L^2}}{\|d_o\|_{L^2}}.$$  \hspace{1cm} (44)
FIG. 7. (Color online) Effect of using several sources and receivers in localizing objects with the drift noise corresponding to 0.001% perturbation in the reference speed of sound; (a) 2, (b) 8, (c) 16, and (d) 24.

FIG. 8. (Color online) Effect of drift compensation in localizing objects with the drift noise corresponding to 2.5% perturbation in the reference speed of sound, using only one source-receiver pair. (a) Without drift. (b) Without drift compensation. (c) With drift compensation.

FIG. 9. (Color online) (a) Test configuration and locations of the objects used in the resolution analysis. (b) The plot of the error vs size of the objects normalized to the wavelength.
The graphs indicate as the objects shrink the strength of the perturbations declines, in turn making different perturbation locations indistinguishable. It should, however, be cautioned that this picture does not provide a necessary condition to identify a limit for a successful localization; for example, two functions with a same energy norm can be quite distinct. Nevertheless, it provides a safe region to achieve satisfactory localization results.

A second measure of resolution can be considered as the minimum distance by which an object (e.g., a training basis) can be offset and yet results in a unique identification of the said change. This limit can be well below a wavelength as long as the object size is in the resolvable regime discussed above. This is by virtue of utilizing the entire reverberant field. As long as an object can create sufficient perturbations, through the action of longtime propagation and reverberation, a set of distinct features can be registered that can suffice to distinguish it from its sub-wavelength neighboring locations.

IX. SUMMARY

This paper presented learning algorithms to address the localization problem in an enclosure. The chief advantages they offer is the capability of reducing the number of spatial measurements by virtue of utilizing the temporal information beyond the classical limit, in both coherent and incoherent phases of propagation. The learning algorithm benefits from the entire reverberant field leading, in turn, to merely a single source-receiver pair. The learning method relies on a prior set of measurements and is constructed based on finding the projection of any arbitrary measurement in the space spanned by the prior set. This is particularly of importance in systems with a limited available knowledge and immense uncertainties. The algorithm calls for the minimum knowledge of the system, and for the most part, looks at it as a black box. Several different improvements of the algorithm were presented based on the motivations from the physics or operational conditions of the system. When the learning algorithm is implemented experimentally, constructing the prior data set can be costly or lack robustness. This motivates to investigate other approaches such as the waveform inversion approach with the aim of removing this dependence as a future direction. There are also other future directions for the proposed methods, such as investigating the mathematical properties as in Sec. VI, more in-depth resolution analysis particularly the effect of bandwidth and noise, or application to a specific field of interest. The methods presented here can be beneficial in different applications such as acoustic source localization in room or underwater acoustic, ultrasonic monitoring of structural defects and damages, ultrasonic touchscreen, and microwave navigation of objects in indoor environments.

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