Localization of weak objects in reverberant fields using waveform inversion

Kamyar Firouzi, and Butrus T. Khuri-Yakub

Citation: The Journal of the Acoustical Society of America 142, 1088 (2017); doi: 10.1121/1.4999047
View online: http://dx.doi.org/10.1121/1.4999047
View Table of Contents: http://asa.scitation.org/toc/jas/142/2
Published by the Acoustical Society of America

Articles you may be interested in

Reciprocity relationships in vector acoustics and their application to vector field calculations
The Journal of the Acoustical Society of America 142, 523 (2017); 10.1121/1.4996458

Beamforming using subspace estimation from a diagonally averaged sample covariance
The Journal of the Acoustical Society of America 142, 473 (2017); 10.1121/1.4995993

Space-time domain solutions of the wave equation by a non-singular boundary integral method and Fourier transform
The Journal of the Acoustical Society of America 142, 697 (2017); 10.1121/1.4996860

Removal of incoherent noise from an averaged cross-spectral matrix
The Journal of the Acoustical Society of America 142, 846 (2017); 10.1121/1.4997923

Two-dimensional grid-free compressive beamforming
The Journal of the Acoustical Society of America 142, 618 (2017); 10.1121/1.4996460

Rigid and elastic acoustic scattering signal separation for underwater target
The Journal of the Acoustical Society of America 142, 653 (2017); 10.1121/1.4996127
Localization of weak objects in reverberant fields using waveform inversion

Kamyar Firouzi(a) and Butrus T. Khuri-Yakub
E. L. Ginzton Laboratory, Stanford University, 348 Via Pueblo Mall, Stanford, California 94305, USA

(Received 6 January 2017; revised 13 July 2017; accepted 30 July 2017; published online 23 August 2017)

This paper presents an application of the Waveform inversion approach to localization of objects in reverberant fields and with limited spatial measurements. Reverberant fields in enclosures can potentially carry useful information, however, in an incoherent way. Incoherency comes from the consecutive reflections of the wave energy several times in the domain. This, along with diffraction and dispersion effects, can ultimately lead to mixing of the wave energy in a seemingly random way. However, spreading of the wave energy can lead to multiple interrogations of each point in the enclosure. Hence, any substructural changes in the enclosure can be sensed with sufficient information carried by the wave energy flow. Furthermore, the temporal information buried in the data makes it feasible to conduct only a few spatial measurements. The authors present a localization scheme that benefits from the reverberant field and can reduce the required number of spatial measurements. © 2017 Acoustical Society of America. [http://dx.doi.org/10.1121/1.4999047]

I. INTRODUCTION

Wave propagation in complex structures in the high-frequency (small wavelength) limit is rich, complicated, and yet of practical use. The complexity of wave media can be either due to the presence of subwavelength inhomogeneities (scattering objects) of the wave velocity, or to the geometry boundaries enclosing a homogeneous medium. It is the latter case that is of interest in this study.

Wave propagation in enclosures is of great practice in a broad range of applications such as room acoustics,1,2 ultrasonic monitoring of structural defects and damages,3,4 ultrasonic touchscreen,5,6 microwave navigation of objects in indoor environments,7 detection of scatterers embedded in reciprocal media,8 etc., all of which share in common an underlying physical nature, the reverberant field.

Reverberation is the process of formation of a wave field in enclosures as a result of a large number of reflections.1 It leads to mixing of the wave energy, which in turn results in incoherent spreading of information. However, spreading of the wave energy can lead to multiple interrogations of each point in the enclosure, which is reminiscent of an ergodic or a closely ergodic behavior.9,10 Hence, any substructural changes in the enclosure can be sensed with sufficient information carried by the wave energy flow.

Incoherent propagation and information mixing in enclosures make the localization problem very challenging. In classical imaging techniques, a set of measurements are collected by a selective combination of sources and receivers of the wave energy (e.g., transducers). The methods heavily rely on the first arrivals of the perturbations and also the compactness of the collected information in time, where the primaries and reflections can be distinct, and higher orders of information are progressively of less practical use. In a reverberant field, on the other hand, the information of the field-perturbing object(s) can be well-mixed. Reflections may not be distinct and can even potentially contaminate the first arrival; and in turn, this would degrade the performance of the imaging algorithm. Furthermore, exploring the reverberant data gives the feasibility of reducing the number of spatial measurements. Spatial measurements in some applications can be costly in several different ways including acquisition time, computational processing, and equipment.

In many classical imaging techniques it is assumed that the domain being imaged is semi-infinite and non-compact, and the radiation condition is in effect, implying there is no physical boundaries of which the waves can reflect. Even in bounded domains the transducers are arranged such that the higher order reflections can be muted. This is a base theme of many imaging technologies such as ultrasound, radar, and seismic imaging techniques.11–13 Naive applications of these methods to reverberant data produce meaningless results.

In reverberant fields, training or learning have widely been applied in different contexts such as localization and classification of defects and flaws in solid substrates,14 localization of tactile objects in contact with a plate,5,6,15,16 and localization by probabilistic pattern recognition.17 In training methods, generally the system is looked at as a black box and any (a posteriori) measured data is matched with a set of a priori measured data.

In the present paper, we study the localization of object(s) in bounded domains using the so-called adjoint state method. The main goal of this study is to apply and assess the adjoint method to localization of weak and finite objects (i.e., objects with characteristic dimensions on the order of a wavelength with properties close to those of the surrounding medium) in a moderately reverberant field. The objects are assumed to be compactly distributed in a bounded domain. We formulate this problem as an inverse problem. The adjoint state method is a mathematical technique designed to efficiently calculate

---

(a)Electronic mail: kfirouzi@stanford.edu
the gradient of an objective functional with respect to the parameter function. It has diverse applications in inverse problems particularly seismic imaging.\textsuperscript{18,19} This, when coupled with the linearization of the objective functional, leads to the \textit{linearized waveform inversion} (LWI). We consider the localization of passive objects; that is, the waves are generated and received by a set of transducers, and the objects perturb the wave field upon interfacing with the wave energy flow.

The organization of the paper is then as follows: In Sec. II, we present a canonical model that governs the physics of the system of interest, which can be universal in interpretation. We next pose the localization problem as an inverse problem in Sec. III. In Sec. IV, we present the mathematical basis of the adjoint method. Following this, we present the linearization theory in Sec. V. The LWI algorithm is presented in Sec. VI. Model misfit compensation is addressed in Sec. VII. In Sec. VIII, we present the evaluation of the method on synthetic data and a summary concludes the paper.

II. MODEL PROBLEM

Physics underlying imaging problems are usually governed by wave propagation or its variants. They are represented by a set of partial differential equations with real characteristic curves (i.e., space-time curves along which information propagates). Let $\partial_t$ and $\partial_n$, respectively, indicate the first and second partial derivatives in time. The most general linear form of this type can be written as

$$\{A(m)\partial_t + B(n)\partial_n + K\} \Phi = F,$$

where $m$ and $n$ are spatial functions that can parametrize the propagation, such as a combination of speed of sound, density, absorption coefficients, etc. $\Phi$ is the wave field such as the acoustic pressure field in rooms or indoor environments or the displacement field in solid substrates. $F$ is the source field by an acoustic source, ultrasonic transducer, or antenna. $A$ and $B$ are generally considered as linear operators of $m$ and $n$, $K$ is a symmetric positive definite operator that is symmetric positive definite.\textsuperscript{20} Two canonical examples are the wave and plate equations\textsuperscript{20,21} or any other form that can be derived from a system of hyperbolic equations. For practical purposes, $K$ is discretized using a numerical scheme such as finite differences or finite elements,\textsuperscript{21,22} whereby $K$ becomes a symmetric positive definite matrix. The boundary conditions are effectively encapsulated inside the discrete representations of $K$ and $F$.

In the present paper, the methods are evaluated for a wave equation of the following type to study and analyze the proposed methods:

$$\left(\frac{\rho(x)}{c_0^2}\partial_t + \sigma(x)\partial_n + \Delta\right)\varphi = f(t)\delta_{\epsilon}(x), \quad \text{in } \Omega,$$

$$\varphi = 0, \quad \text{on } \partial \Omega,$$

$$\varphi(\cdot, t = 0)$$

in which $\Omega$ represents the wave domain (with $\partial \Omega$ being its boundary) and $c_0$ is the speed of propagation with some spatial perturbations represented by $\rho$. $\sigma$ is the effective absorption. $\delta_{\epsilon}$ is the Dirac delta, representing a point source in space. Upon discretization, it gives the following representation:

$$\{A(m)\partial_t + B(n)\partial_n + K\} \Phi = F,$$

$$A(m) = \frac{1}{c_0^2} m, \quad B(n)$$

Nonetheless, the methods will be presented and derived based on Eq. (1), which represents the localization problem in wave enclosures in its full generality and the operators $A$, $B$, and $K$ can be adjusted to the context of interest. Moreover, it will be shown later that the final algorithms are independent of these operators.

III. LOCALIZATION AS AN INVERSE PROBLEM

Localization of an object can be formulated as an inverse problem. Let $\mathcal{L}$ be the (nonlinear) operator that maps the parameter functions to the measured data $d$. Let $\mathcal{D}$ be the data space spanned by the measurements corresponding to each source-receiver pair. The localization problem can be expressed then as

$$\hat{m}, \hat{n} = \arg\min_{m, n} \frac{1}{2} \left\| \mathcal{L}(m, n) - d \right\|^2_{L^2(D \times [0, T])}.$$

This poses the challenge of estimating the operator $\mathcal{L}$, which is inherently nonlinear. Among literature, the most common techniques include the Continuum finite differences or finite elements, and boundary integral methods. The former is predominant in seismic imaging\textsuperscript{12} where a real-time implementation is not crucial. The latter is in practice in radar and ultrasound\textsuperscript{11} imaging, where only imaging primary echoes is of concern.

In practice, the wave field $\Phi$ is not measured at every point in space, but only at a subset of points. This process can be identified by defining a \textit{down sampling} operator $\mathcal{P}$, i.e., a matrix whose elements are either 1 if the corresponding point is being observed or 0 otherwise. Using the model problem, the inversion can be reformulated as

$$\min_{m, n} \frac{1}{2} \left\| \mathcal{P}\Phi - d \right\|^2_{L^2(D \times [0, T])},$$

subject to

$$\{A(m)\partial_t + B(n)\partial_n + K\} \Phi = F.$$

Generally, in order to solve such an optimization problem, the gradient of the objective functional with respect to the parameter functions is required. One possible approach is to construct a set of grid points for the parameter functions. For each grid point, the constraint is solved in order to calculate a corresponding field value, and hence, the corresponding value of the objective functional. The functional gradient can next be approximated by means of a finite differencing scheme. This approach can be potentially expensive, in practice. The \textit{adjoint state} method is an alternative scheme meant to calculate the gradient of the objective functional in an efficient way.\textsuperscript{19}
IV. ADJOINT STATE METHOD

Consider the model problem, Eq. (1), in which we are interested to reconstruct the parameters \( m \) and \( n \). Let
\[
L := \{ A(m) \partial_t + B(n) \partial_l + K \}.
\]
We can then write
\[
L \Phi = \mathbf{F}.
\]
Note also that the equation above can be rearranged as
\[
L = \Sigma^* \left\{ \frac{\partial_v}{\partial_t} \right\} + K, \quad \Sigma = \left\{ \begin{array}{c} A(m) \\ B(n) \end{array} \right\},
\]
where \( \dagger \) represents adjoint. Therefore, the optimization problem (5) can be rewritten as
\[
\min_{\Sigma} J(\Sigma) = \frac{1}{2} \int_0^T \| P \Phi - d \|_{L^2(\Omega)}^2 \, dt,
\]
subject to
\[
L(\Sigma) \Phi = \mathbf{F},
\]
The gradient descent method can now be stated as
\[
\Sigma_{i+1} = \Sigma_i + \gamma' \frac{\partial J}{\partial \Sigma} \bigg|_{\Sigma_i},
\]
with a scaling parameter \( \gamma' \in \mathbb{R} \) for the \( i \)th iteration. Hence, at least it is required to calculate the gradient of the objective functional with respect to the parameter function \( \Sigma \). Note that this is still an optimization problem constrained to a time-dependent set of differential equations. The adjoint state method is designed to provide an efficient way of solving such optimization problems. The adjoint field indicated by \( \lambda \) is interpreted as the backward propagating field. For more details and full derivation see the Appendix. The final algorithm reads as follows. Starting with an initial guess \( \Sigma_o \),
1. calculate the forward field using the source term \( \mathbf{F} \):
\[
L(\Sigma) \Phi = \mathbf{F}, \quad \Phi(\cdot, t = 0) = \partial_t \Phi(\cdot, t = T) = 0,
\]
2. solve the adjoint equation with the measurement residuals as the source terms (the back-propagation step):
\[
L^* \left( \lambda \right) = -P^* (P \Phi - d),
\]
\[
\lambda(\cdot, t = T) = \partial_t \lambda(\cdot, t = T) = 0,
\]
3. calculate the gradient of the objective functional and update the parameter function:
\[
\Sigma_{i+1} = \Sigma_i + \gamma' \int_0^T \langle \lambda', \Phi(\cdot, t) \rangle \, dt,
\]
4. repeat until convergence.

Since the problem of interest deals with a reverberant field, as we will show later, increasing the timespan of the measured data may not always result in a more accurate reconstruction. This motivates to improve the final step by computing the maximum time-correlation between the forward and backward fields, i.e.,
\[
\Sigma_{i+1} = \Sigma_i + \gamma' \max_{\tau \in [0,T]} \int_0^T \langle \lambda', \tau - t \rangle, \Psi(\cdot, t) \rangle \, dt.
\]
The process above may be referred to as the full wave-form inversion \(^{23}\) or least squares migration \(^{19}\), which may have a slow rate of convergence \(^{23}\) and can be sensitive to the initial guess, in particular when applied to the problem at hand. Furthermore, it can be computationally intense for real-time applications. In Sec. V, we present a linearization that targets the application to the problem of interest in this study.

V. LINEARIZATION

As emphasized before, the relation between the parameter function \( \Sigma \) and the field function \( \Phi \) is nonlinear. In this section, we attempt to construct an appropriate linear relation between the field and the parameter. Consider a base state \( \Phi_o \), in response to a base parameter function \( \Sigma_o \), i.e., a field with no perturbation in the parameter function; that is, the base state satisfies
\[
L(\Sigma_o) \Phi_o = \mathbf{F},
\]
Imposing the perturbation \( \delta \Sigma \) over the parameter \( \Sigma_o \) would consequently induce the perturbation \( \delta \Phi \) over the wave field \( \Phi_o \). The first order perturbation theory can be used to determine the correction \( \delta \Phi \) to the base wave field in response to \( \delta \Sigma \); that is, assume
\[
\Phi = \Phi_o + \delta \Phi,
\]
\[
\Sigma = \Sigma_o + \delta \Sigma,
\]
then replace these expressions in the model problem:
\[
L(\Sigma_o + \delta \Sigma) \{ \Phi + \delta \Phi \} = \mathbf{F},
\]
from which if
\[
\| \delta \Sigma \| \| \delta \Phi \| \| \Phi_o \| \ll 1,
\]
we obtain
\[
L_o \delta \Phi = - \delta L \Phi_o.
\]
Note that the relation between the parameter perturbation and the field perturbation is now linear. This can be viewed as a Born approximation. Then upon down sampling the field at the measurement points, we have
\[
\delta d = - P L_o^{-1} \delta \Phi_o,
\]
where the negative exponent indicates the operator inverse. In order to define a linear map between the parameter and measurements, it is essential to rearrange the equation above as

1090 J. Acoust. Soc. Am. 142 (2), August 2017
Kamyar Firouzi and Butrus T. Khuri-Yakub
\[
\delta \Phi_o = \begin{pmatrix}
\text{diag}(\partial_o \Phi_o) & 0 \\
0 & \text{diag}(\partial_o \Phi_o)
\end{pmatrix} \delta \Sigma,
\]

which then leads to
\[
\delta d = M \delta \Sigma, \quad M = -PL_o^{-1} \delta \Phi_o.
\] (20)

Hence, the data model can be expressed as a linear map from the parameters, through the operator \( M \). The performance of the linear approximation for a two-dimensional rectangular domain is shown in Fig. 1 for weak and strong objects, corresponding to 1% and 10% perturbations with respect to the background speed, respectively. The simulation setting is the one described in Sec. VIII, with the source and receiver placed at two opposite edges, 100 mm apart, and with a 3 mm lateral offset. As expected, the strength of the perturbations is inversely proportional to the accuracy of the linear approximation.

A. Higher orders and break down of the Born approximation in a longtime reverberant field

The approximation above can be successively advanced in order to obtain a better approximation, with hopefully a convergent behavior in the limit. For this, starting from the base state, we compute

\[
L_o \delta \Phi_1 = -\delta L \Phi_o,
\]
(21a)

\[
\Phi_1 = \Phi_o + \delta \Phi_1,
\]
(21b)

and then for \( i \geq 1 
\]

\[
L_o \delta \Phi_{i+1} = -\delta L \delta \Phi_i,
\]
(22a)

\[
\Phi_{i+1} = \Phi_i + \delta \Phi_{i+1}.
\]
(22b)

This approximation potentially converges to the exact solution. To see this, note

\[
\delta \Phi_{i+1} = -L_o^{-1} \delta L \delta \Phi_i = (-L_o^{-1} \delta L)^{i+1} \Phi_o.
\]
(23)

Hence

\[
\Phi_{i+1} = \Phi_o + \sum_{k=0}^{i+1} \delta \Phi_k = \sum_{k=0}^{i+1} (-L_o^{-1} \delta L)^k \Phi_o.
\]
(24)

Next, let \( i \rightarrow \infty \). Then

\[
\Phi_\infty = -(I + L_o^{-1} \delta L)^{-1} \Phi_o = (I + L_o^{-1} \delta L)^{-1} L_o^{-1} F = (L_o + \delta L)^{-1} F = \Phi.
\]
(25)

This approximation, however, can be unstable. As an example see Fig. 2, where the error of the perturbation solution versus the number of iterations was calculated. In some cases, it can be stabilized by applying a frequency low-pass filter at each iteration to attenuate all the frequencies higher than the highest frequency present in the source.

The Born approximation as presented above can be viewed as a regular perturbation approach to the model problem. This approximation breaks down in a longtime behavior since Eq. (17) fails to be valid; namely, \( \| \delta \Phi \| / \| \Phi_o \| \) grows and becomes of \( O(1) \), while \( \| \delta \Sigma \| \) is kept fixed. This is shown in Fig. 2 by progressively adding a longer time-window to the calculation of the error of the linear model. This is a consequence of the secular behavior of the regular perturbation to problems of this type.\(^{24}\) The remedy is a singular perturbation theory known as the method of multiple timescales. Essentially in the long run, the system should be assumed to have dynamical behaviors at two different timescales, the so-called slow and fast times.

Nonetheless, it will be shown later that the linear approximation can still be utilized to localize objects using early-time reverberation data, which include the primary and several reflections inside the domain. Exploring the reverberant data in turn gives the feasibility of reducing the number of spatial measurements.

VI. LWI

In the preceding section, we showed that upon linearization

\[
\delta d = M \delta \Sigma, \quad M = -PL_o^{-1} \delta \Phi_o.
\]
lem is given by the normal equation, i.e.,

\[ \Sigma = M^T M \delta d. \]  

and the general convention is to neglect the so-called Hessian, \( H = (M^T M)^{-1} \), which is ill-conditioned and difficult to compute. This in turn gives

\[ \delta \Sigma \approx M^T \delta d. \]  

In fact, we can now show, upon neglecting the Hessian, solution (28) corresponds to performing merely one iteration of the gradient descent algorithm as in Sec. IV. For this, we can recast the gradient descent algorithm as

\[ \delta \Sigma = \delta \Sigma_o + \gamma \int_0^T \langle \lambda^o(\cdot, t), \Psi_o(\cdot, t) \rangle dt, \]  

and set the initial guess \( \delta \Sigma_o = 0 \). Consequently,

\[ \delta \Sigma \approx \int_0^T \langle \lambda^o(\cdot, t), \Psi_o(\cdot, t) \rangle dt. \]  

Equations (28) and (30) are identical. To see this, we apply the gradient descent method to the linearized least squares problem, with \( \gamma' = 1 \), for all \( i \), which leads to

\[ \delta \Sigma_{i+1} = \delta \Sigma_i - M^T (M \delta \Sigma_i - \delta d) \]

\[ = (I - H) \delta \Sigma_i - M^T \delta d \]

\[ = \sum_{k=0}^i (I - H)^{k+1} \delta \Sigma_o - (I - H)^k M^T \delta d. \]  

Now, set \( \delta \Sigma_o = 0 \) and let \( i \to \infty \). Then

\[ \delta \Sigma_{\infty} = \left( \sum_{k=0}^\infty (I - H)^k \right) M^T \delta d \]

\[ = H^{-1} M^T \delta d, \]  

which is the exact least squares solution. Note that in the last step, we used the Neumann expansion of the operator \((I - H)^{-1}\), assuming non-singularity.

In summary, the linearized localization algorithm is as follows: Starting with a base state \( \Sigma_o \), i.e., in the absence of any field-perturbing object:

1. calculate the forward field using the source term \( F \):

\[ L(\Sigma_o) \Phi_o = F, \quad \Phi_o(\cdot, t = 0) = \partial_t \Phi_o(\cdot, t = T) = 0, \]  

2. solve the adjoint equation backward in time with the measurement residuals as the source terms:

\[ L^\dagger(\Sigma_o) \lambda = -P^\dagger (P \Phi_o(d) - d), \]

\[ \lambda(\cdot, t = T) = \partial_t \lambda(\cdot, t = T) = 0, \]  

3. calculate the time-correlation of the forward and backward fields:

\[ \delta \Sigma \approx \max_{\tau \in [0,T]} \int_0^T \langle \lambda(\cdot, \tau - t), \Phi_o(\cdot, t) \rangle dt. \]  

VII. MODEL MISFIT COMPENSATION

As pointed out earlier, the success of the linearized localization method relies on how well \( M \Sigma_o = d_o \). This calls for a well-developed data model. In the problem of interest in this study, the propagation model can be somewhat challenging to construct in practical applications. The chief challenge is a proper construction of a set of boundary conditions. In real applications, there exist many sources of uncertainties in the boundary conditions, material properties, and dispersion of the waves. Hence, the underlying physics of the measured data might have offsets from the model. On
one hand, the material properties appear as the base parameter function \( \Sigma_o \). On the other hand, the boundary conditions and dispersion appear as uncertainties in the operator \( K \). Dispersion can somewhat easily be managed, because it relies on the propagation in an infinite domain. To take care of the boundary condition uncertainties, observe that any boundary condition can be written as a linear combination of the homogeneous boundary conditions of the first and second kinds. This motivates to decompose the operator \( K \) into two pieces as \( K = aK_a + bK_b \), where \( K_a \) and \( K_b \) are, respectively, associated with the homogeneous boundary conditions of the first and second kinds. We can now use the adjoint state method to calibrate the propagation model, by solving for the optimal \( \Sigma_o, a, \) and \( b \),

\[
\min_{\Sigma_o,x,\beta} \frac{1}{2} \int_0^T \| P\Phi - d_s \|_{L^2(\Omega)}^2 dt, \tag{36a}
\]

subject to

\[
L(\Sigma_o, x, \beta) \Phi_o = F. \tag{36c}
\]

We also require \( \Sigma_o, x, \) and \( \beta \) to be single numbers; they can, however, be considered as fields. The initial guess can be estimated based upon the best available prior knowledge about the system, e.g., the nominal Young’s modulus and Poisson ratios of the domain.

**VIII. NUMERICAL TESTS**

In order to verify the waveform inversion algorithm, the model problem, Eq. (3), was discretized using a fourth-order finite difference scheme in space with a second order two-step time integration.\(^{27}\) In the model, we set \( c_0 = 4000 \text{ m/s}, \ m_o = 1, \ n_o = 0.001. \) The reverberant domain is considered as a two-dimensional 100 mm \( \times \) 60 mm rectangular domain with the sources and receivers evenly placed at the boundaries. The sources and receivers are assumed collocated. The number of
sources and receivers was varied from 4 to 16 and the timespan of data from 75 to 300 $\mu$sec in each of the case studies presented below. The input source profile is a Gaussian-modulated sine wave with a center frequency at 900 kHz with a 35% bandwidth. Ten points per minimum wavelength with Courant-Friedrichs-Lewy (CFL) criterion $\frac{1}{2}$ were used.

The field-perturbing objects have circular intersections with a radius of 2.5 mm (on the order of a wavelength) and with about 1% perturbation over the base parameters. We consider a few arrangements of one, three, and five objects randomly placed in the domain. The arrangements, however, were kept fixed among the test cases for the sake of comparison. The arrangements are shown in Fig. 3. In simulating the synthetic data, we set $\delta m_s = 0.01$, $\delta n_a = 0.00001$. The localization results using 100 $\mu$sec and 8 source and receiver pairs are shown in Fig. 4, where, compared to the learning theory, a small timespan of data, however at more sources and receivers, was utilized.

The effect of the timespan of data is depicted in Fig. 5 using eight source and receiver pairs, where progressively larger time-windows of information were used in the localization process. It can be observed that up to a certain point (100 $\mu$sec), adding more data helps improve the localization result, however, beyond this point, the localization result degrades. This is due to the underlying linearized model since, as explained in Sec. VA, it is valid up to the early-time reverberant field.

In a similar set of tests, the effect of a different number of sources and receivers are depicted in Fig. 6 using 100 $\mu$sec of data, where progressively more source and receiver pairs were employed in the localization process. The results suggest adding more sources and receivers helps to improve the contrast of the localized object. Note that even for the case of 16 transducers, the minimum distance between the transducers is $\approx 20$ mm, about 5 wavelengths, indicating a sparse configuration of the transducers with respect to the wavelength.

IX. SUMMARY

This paper investigated the application of the LWI algorithm to the localization problem in an enclosure. The chief advantage it offers is the capability of reducing the number of spatial measurements by virtue of utilizing the temporal...
information beyond the classical limit, in both coherent and incoherent phases of propagation.

The algorithm herein aims at localizing objects in an early reverberant field. Field reverberations manifest as a result of several reflections of the wave energy off the boundaries, which in turn leads to mixing of information in a seemingly chaotic way. This, however, potentiates utilizing the incoherent information in inferring any sub-structural changes appearing because of the presence of field-perturbing objects.

The waveform inversion logarithm is constructed based upon the linearized data model. The linearization herein is a regular perturbation approach to this problem and is valid only up to the early-time reverberant field; it starts to fail as sound waves become diffusive. An immediate generalization of this model is the singular perturbation approach, which would result in a more accurate multi-scale model. The multi-scale model can be further simplified to obtain a sound diffusion model governing the slow timescale of the problem.

The performance of the waveform inversion algorithm heavily relies on the propagation model, which in practice carries diverse sources of uncertainties such as the ones in the boundary conditions, alignment of the transducers, dispersion, and mechanical or environmental effects. Among these factors, the boundary conditions appeared as the most crucial feature that must be the target of future works.

APPENDIX: DERIVATION OF THE ADJOINT STATE METHOD

The adjoint state method is designed to provide an efficient way of solving such optimization problems. It can be derived by means of several techniques; we present the one based on the Lagrange multiplier method. Consider \( \lambda \) as the Lagrange multiplier (or the adjoint state, for a reason that becomes apparent shortly), and define the augmented objective functional

\[
\tilde{J}(\Sigma, \Phi, \lambda) := \frac{1}{2} \int_0^T \left\{ \| P \Phi - d \|_{L^2(D)}^2 + \langle \lambda, L\Phi - F \rangle \right\} dt,
\]

(A1)
is precisely what is required to initiate the gradient descent method. The critical point in the derivation is to perform a few integration parts on the terms that involve the multiplier

\[
\int_0^T \langle \lambda, L\Phi \rangle dt = \int_0^T \langle \lambda, \{A(m)\partial_r + B(n)\partial_t + K\} \Phi \rangle dt
\]

\[
= \int_0^T \langle \Phi, \{A(m)\partial_r - B(n)\partial_t + K\} \lambda \rangle dt
\]

\[
+ \langle \lambda, A(m)\partial_r \Phi \rangle |_0^T - \langle \partial_t \lambda, A(m)\Phi \rangle |_0^T
\]

\[- \langle \lambda, B(n)\Phi \rangle |_T.
\]

(A2)

In the model problem, we assumed \(\Phi(\cdot, t = 0) = \partial_t \Phi(\cdot, t = 0) = 0\). Hence,

\[
\int_0^T \langle \lambda, L\Phi \rangle dt = \int_0^T \langle \Phi, L^\dagger \lambda \rangle dt
\]

\[
+ \langle \lambda, A(m)\partial_r \Phi \rangle |_0^T - \langle \partial_t \lambda, A(m)\Phi \rangle |_T
\]

\[- \langle \lambda, B(n)\Phi \rangle |_T.
\]

(A3)

where \(L^\dagger = \{A(m)\partial_r - B(n)\partial_t + K\}\), the adjoint wave operator. Next observe that

\[
\frac{\partial \tilde{J}}{\partial \Phi} = \int_0^T \{P(\partial_t \Phi - d) - L^\dagger \lambda \} dt.
\]

(A4)

The basic idea behind the adjoint state method is to construct \(\lambda\) such that \(\frac{\partial \tilde{J}}{\partial \Phi} = 0\), for all \(t\), which suggests that \(\lambda\) should satisfy

\[
L^\dagger \lambda = -P(\partial_t \Phi - d), \quad \lambda(\cdot, t = T) = \partial_t \lambda(\cdot, t = T) = 0.
\]

(A5)

Also, observe that by construction

\[
\frac{\partial \tilde{J}}{\partial \lambda} = \int_0^T \{L\Phi - F\} dt = 0.
\]

(A6)

Consequently

\[
\frac{\partial \tilde{J}}{\partial \Sigma} = \int_0^T \langle \lambda, \Psi \rangle dt,
\]

(A7)

where

\[
\Psi = \left\{ \frac{\partial_t \Phi}{\partial \Phi} \right\}.
\]

Hence, we constructed \(\lambda\) such that

\[
\frac{\partial \tilde{J}}{\partial \Sigma} = \frac{\partial \tilde{J}}{\partial \Sigma} = \int_0^T \langle \lambda, \Psi \rangle dt.
\]

(A8)

Note that \(\lambda\) satisfies the adjoint model problem, but with the initial conditions prescribed at the final time \(t = T\). Hence, the adjoint problem should be solved backward in time, or equivalently, with respect to a new time \(t' = T - t\) from 0 to \(T\). We can now construct the gradient descent method as follows. Starting with an initial guess \(\Sigma_0\),

(1) calculate the forward field using the source term \(F\):

\[
L(\Sigma_k) \Phi_k = F, \quad \Phi_k(\cdot, t = 0) = \partial_t \Phi_k(\cdot, t = T) = 0, \quad (A9)
\]

(2) solve the adjoint equation with the measurement residuals as the source terms (the back-propagation step):

\[
L^\dagger(\Sigma_k) \lambda_k = -P^\dagger(P \Phi_k - d), \quad \lambda_k(\cdot, t = T) = \partial_t \lambda_k(\cdot, t = T) = 0, \quad (A10)
\]

(3) calculate the gradient of the objective functional and update the parameter function:

\[
\Sigma_{k+1} = \Sigma_k + \gamma \int_0^T \langle \lambda_k(\cdot, T - t), \Psi(\cdot, t) \rangle dt, \quad (A11)
\]

(4) repeat until convergence.


