AA216/CME345: PROJECTION-BASED MODEL ORDER REDUCTION

Moment Matching Methods (M^3)

Charbel Farhat
Stanford University
cfarhat@stanford.edu

These slides are based on: the recommended textbook, A.C. Antoulas, "Approximation of Large-Scale Dynamical Systems," Advances in Design and Control, SIAM, ISBN-0-89871-529-6; and on papers co-authored by Hetmaniuk, Tezaur, and Farhat

Outline

$$\boxed{\begin{aligned} \frac{d\mathbf{w}}{dt}(t) &= \mathbf{A}\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{w}(t) + \mathbf{D}\mathbf{u}(t) \\ \mathbf{w}(0) &= \mathbf{w}_0 \end{aligned}}$$

- $\mathbf{w} \in \mathbb{R}^N$: Vector of state variables
- $\mathbf{u} \in \mathbb{R}^{in}$: Vector of input variables – typically $in \ll N$
- $\mathbf{y} \in \mathbb{R}^q$: Vector of output variables – typically $q \ll N$

- Goal: Construct a Projection-based Reduced-Order Model (PROM)

$$\boxed{\begin{array}{rcl} \frac{d\mathbf{q}}{dt}(t) & = & \mathbf{A}_r\mathbf{q}(t) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}(t) & = & \mathbf{C}_r\mathbf{q}(t) + \mathbf{D}_r\mathbf{u}(t) \end{array}}$$

  - $\mathbf{q} \in \mathbb{R}^k$: Reduced-order vector of state variables (or vector of generalized coordinates) – $k \ll N$

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  - $\mathbf{q} \in \mathbb{R}^k$: Reduced-order vector of state variables (or vector of generalized coordinates) – $k \ll N$

- For a Petrov-Galerkin PROM

$$\begin{aligned} \mathbf{A}_r &= (\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V} \in \mathbb{R}^{k\times k} \\ \mathbf{B}_r &= (\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B} \in \mathbb{R}^{k\times in} \\ \mathbf{C}_r &= \mathbf{C}\mathbf{V} \in \mathbb{R}^{q\times k} \\ \mathbf{D}_r &= \mathbf{D} \in \mathbb{R}^{q\times in} \end{aligned}$$

- Let $\mathbf{h}$ denote a matrix-valued function of time representing the kernel of an LTI system

$$\mathbf{h}: \ t \in \mathbb{R} \longmapsto \mathbb{R}^{q \times in}$$

Example: Impulse response of an LTI system

$$\mathbf{h}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{B}}_{\mathbf{h}_a(t)} + \underbrace{\mathbf{D}}_{\mathbf{h}_0}\delta(t)$$

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- Let $\mathbf{H}(s) \in \mathbb{R}^{q \times in}$ denote its Laplace transform

$$\mathbf{H}(s) = \int_0^{\infty} e^{-st}\mathbf{h}(t)dt$$

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$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

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- $\mathbf{H}(s)$ is the transfer function associated with the HDM defined by $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ as for each input $\mathbf{U}(s)$, it defines the output

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$$

- Let $m \in \{0,\ 1,\ \cdots,\ \}$
  The m-th **moment** of $\mathbf{h}:\ t \in \mathbb{R} \longmapsto \mathbb{R}^{q \times in}$ at $s_0 \in \mathbb{C}$ is

$$\boxed{\eta_m(s_0) = \int_0^\infty t^m e^{-s_0 t}\mathbf{h}(t)\, dt}$$

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- Hence, the m-th **moment** of $\mathbf{h}$ can be written in terms of the transfer function $\mathbf{H}(s)$ as follows

$$\boxed{\eta_m(s_0) = (-1)^m \left. \frac{d^m \mathbf{H}}{ds^m}(s) \right|_{s=s_0}}$$

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Example: Impulse response of an LTI system

$$\begin{aligned}
\eta_0(s_0) &= \mathbf{H}(s_0) = \mathbf{C}(s_0 \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\
\eta_m(s_0) &= m!\ \mathbf{C}(s_0 \mathbf{I}_N - \mathbf{A})^{-(m+1)}\mathbf{B}, \quad \forall m \geq 1
\end{aligned}$$

- Development of $\mathbf{H}(s)$ in Taylor series

$$
\begin{aligned}
\mathbf{H}(s) &= \mathbf{H}(s_0) + \left.\frac{d\mathbf{H}}{ds}(s)\right|_{s=s_0} \frac{(s-s_0)}{1!} + \cdots \\
&\quad + \left.\frac{d^m\mathbf{H}}{ds^m}(s)\right|_{s=s_0} \frac{(s-s_0)^m}{m!} + \cdots \\
&= \eta_0(s_0) - \eta_1(s_0)\frac{(s-s_0)}{1!} + \cdots + (-1)^m \eta_m(s_0)\frac{(s-s_0)^m}{m!} + \cdots \\
&= \eta_0(s_0) + \eta_1(s_0)\frac{(s_0-s)}{1!} + \cdots + \eta_m(s_0)\frac{(s_0-s)^m}{m!} + \cdots
\end{aligned}
$$

- The **Markov parameters** of the system defined by **h** are defined as the coefficients $\eta_m(\infty)$ of the expansion in Laurent series of the transfer function at infinity

$$\boxed{\mathbf{H}(s) = \eta_0(\infty) + \frac{1}{s}\eta_1(\infty) + \frac{1}{s^2}\eta_2(\infty) + \cdots + \frac{1}{s^m}\eta_m(\infty) + \cdots}$$

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$$\begin{aligned} \eta_0(\infty) &= \mathbf{D} \\ \eta_m(\infty) &= \mathbf{C}\mathbf{A}^{m-1}\mathbf{B}, \ \forall m \geq 1 \end{aligned}$$

Proof: Use the property that for $s \to \infty$,

$$(s\mathbf{I}_N - \mathbf{A})^{-1} = \frac{1}{s}\mathbf{I}_N + \frac{1}{s^2}\mathbf{A} + \cdots + \frac{1}{s^{m+1}}\mathbf{A}^m + \cdots$$

pre-multiply by **C**, post-multiply by **B**, and identify with the expansion given above

- Let $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ represent the HDM defined by $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and let $s_0 \in \mathbb{C}$

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- *Objective*: Construct a PROM $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ such that the first l moments $\{\eta_{r,j}(s_0)\}_{j=0}^{l-1}$ of its transfer function at s_0, $\mathbf{H}_r = \mathbf{C}_r(s_0\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r + \mathbf{D}_r \in \mathbb{R}^{q \times in}$, match the first l moments $\{\eta_j(s_0)\}_{j=0}^{l-1}$ of the transfer function $\mathbf{H}(s) \in \mathbb{R}^{q \times in}$ of the HDM

$$\boxed{\eta_{r,j}(s_0) = \eta_j(s_0) \Leftrightarrow \mathbf{H}_r^{(j)}(s_0) = \mathbf{H}^{(j)}(s_0), \ \forall j = 0, \cdots, l-1}$$

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  - the *direct* matching of the moments is in general a numerically unstable procedure
  - today, moment matching is best performed using an equivalent procedure based on Krylov subspaces
- For simplicity, focus is set on the Single Input-Single Output (SISO) $(in = q = 1)$ case throughout the remainder of this chapter

$$\mathbf{B} = \mathbf{b} \in \mathbb{R}^N,\ \mathbf{C}^T = \mathbf{c}^T \in \mathbb{R}^N$$

Theorem

Let $\mathbf{V}$ *be a right Reduced-Order Basis (ROB) such that*

$$range(\mathbf{V}) = \mathcal{K}_k(\mathbf{A}, \mathbf{b}) = span\left\{\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{k-1}\mathbf{b}\right\}$$

and $\mathbf{W}$ *be a left ROB satisfying*

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the PROM of dimension k obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ *using* $\mathbf{W}$ *and* $\mathbf{V}$ *satisfies*

$$\eta_{r,j}(\infty) = \eta_j(\infty) \Leftrightarrow \mathbf{H}_r^{(j)}(\infty) = \mathbf{H}^{(j)}(\infty), \ \forall j = 0, \cdots, k-1$$

Definition

The order-k Krylov subspace generated by $\mathbf{A} \in \mathbb{R}^{N \times N}$ and $\mathbf{b} \in \mathbb{R}^N$ is

$$\mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{k-1}\mathbf{b}\}$$

Remark: Constructing $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ requires only the ability to compute the action of the matrix $\mathbf{A}$ on a vector. In many applications, such a computation can be performed without forming explicitly the matrix $\mathbf{A}$.

The following lemma is introduced to prove the previous theorem

Lemma

The moments of the transfer function of a PROM do not depend on the underlying left and right ROBs, but only on the subspaces associated with these ROBs

Proof of the Theorem.

From the above lemma, it follows that $\mathbf{V}$ can be chosen as follows

$$\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_i \ \cdots \ \mathbf{v}_k] = [\mathbf{b} \ \mathbf{A}\mathbf{b} \ \cdots \ \mathbf{A}^{i-1}\mathbf{b} \ \cdots, \ \mathbf{A}^{k-1}\mathbf{b}]$$

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}_k \Rightarrow \mathbf{A}\mathbf{V}\mathbf{W}^T\mathbf{v}_i = \mathbf{A}\mathbf{V}\mathbf{e}_i = \mathbf{A}\mathbf{v}_i = \mathbf{v}_{i+1} = \mathbf{A}^i\mathbf{b}$$

$$\begin{aligned}
\Rightarrow \quad \eta_{r,0}(\infty) &= \mathbf{D} = \eta_0(\infty) \\
\eta_{r,1}(\infty) &= \mathbf{c}_r\mathbf{b}_r = \mathbf{c}\mathbf{V}\mathbf{W}^T\mathbf{b} = \mathbf{c}\mathbf{V}\mathbf{W}^T\mathbf{v}_1 = \mathbf{c}\mathbf{V}\mathbf{e}_1 = \mathbf{c}\mathbf{b} = \eta_1(\infty) \\
\eta_{r,j+1}(\infty) &= \mathbf{c}_r\mathbf{A}_r^j\mathbf{b}_r = \mathbf{c}\mathbf{V}\mathbf{W}^T(\mathbf{A}\mathbf{V}\mathbf{W}^T)^j\mathbf{b} = \mathbf{c}\mathbf{V}\mathbf{W}^T(\mathbf{A}\mathbf{V}\mathbf{W}^T)^j\mathbf{v}_1 \\
&= \mathbf{c}\mathbf{V}\mathbf{W}^T\mathbf{v}_{j+1} = \mathbf{c}\mathbf{V}\mathbf{e}_{j+1} = \mathbf{c}\mathbf{v}_{j+1} = \mathbf{c}\mathbf{A}^j\mathbf{b} = \eta_{j+1}(\infty)
\end{aligned}$$

Theorem

Let $s_0 \in \mathbb{C}$, $\mathbf{V}$ *be a right ROB satisfying*

$$\begin{aligned} range(\mathbf{V}) &= \mathcal{K}_k\left((s_0\mathbf{I}_N - \mathbf{A})^{-1}, (s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right) \\ &= span\left\{(s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (s_0\mathbf{I}_N - \mathbf{A})^{-k}\mathbf{b}\right\} \end{aligned}$$

and $\mathbf{W}$ *be a left ROB satisfying*

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the PROM of dimension k *obtained by Petrov-Galerkin projection of the HDM* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ *using* $\mathbf{W}$ *and* $\mathbf{V}$ *satisfies*

$$\eta_{r,j}(s_0) = \eta_j(s_0) \Leftrightarrow \mathbf{H}_r^{(j)}(s_0) = \mathbf{H}^{(j)}(s_0), \ \forall j = 0, \cdots, k-1$$

and therefore is an *interpolatory* *PROM*

- This is a more computationally expensive procedure as the computation of each Krylov basis vector requires the solution of a large-scale system of equations

Theorem

Let $s_i \in \mathbb{C}$, $i = 1, \cdots, k$, $\mathbf{V}$ be a right ROB satisfying

$$range(\mathbf{V}) = span\left\{(s_1\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (s_k\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right\}$$

and $\mathbf{W}$ be a left ROB satisfying

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the PROM of dimension k obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ using $\mathbf{W}$ and $\mathbf{V}$ satisfies

$$\eta_{r,0}(s_i) = \eta_0(s_i) \Leftrightarrow \mathbf{H}_r(s_i) = \mathbf{H}(s_i), \ \forall i = 1, \cdots, k$$

and therefore is an <u>interpolatory</u> PROM

Theorem

Let $s_i \in \mathbb{C}$, $i = 1, \cdots, l$, $\mathbf{V}$ *be a right ROB satisfying*

$$range(\mathbf{V}) = \bigcup_{i=1}^{l} \mathcal{K}_k\left((s_i\mathbf{I}_N - \mathbf{A})^{-1}, (s_i\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right)$$

and $\mathbf{W}$ *be a left ROB satisfying*

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the PROM of dimension lk obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ *using* $\mathbf{W}$ *and* $\mathbf{V}$ *satisfies*

$$\eta_{r,j}(s_i) = \eta_j(s_i) \Leftrightarrow \mathbf{H}_r^{(j)}(s_i) = \mathbf{H}^{(j)}(s_i), \ \forall i = 1, \cdots, l, \ \forall j = 0, \cdots, k-1$$

and therefore is an interpolatory PROM

Theorem

Let $s_i \in \mathbb{C}$, $i = 1, \cdots, 2l$, $\mathbf{V}$ be a right ROB satisfying

$$range(\mathbf{V}) = \bigcup_{i=1}^{l} \mathcal{K}_k \left((s_i \mathbf{I}_N - \mathbf{A})^{-1}, (s_i \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right)$$

and $\mathbf{W}$ be a left ROB satisfying

$$range(\mathbf{W}) = \bigcup_{i=l+1}^{2l} \mathcal{K}_k \left((s_i \mathbf{I}_N - \mathbf{A}^T)^{-1}, (s_i \mathbf{I}_N - \mathbf{A}^T)^{-1}\mathbf{c}^T\right)$$

and $\mathbf{W}^T\mathbf{V}$ is nonsingular

Then, the PROM of dimension $2lk$ obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ using $\mathbf{W}$ and $\mathbf{V}$ satisfies

$$\eta_{r,j}(s_i) = \eta_j(s_i) \Leftrightarrow \mathbf{H}_r^{(j)}(s_i) = \mathbf{H}^{(j)}(s_i), \ \forall i = 1, \cdots, 2l, \ \forall j = 0, \cdots, k-1$$

and therefore is an <u>interpolatory</u> PROM

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- Rational interpolation requires the following Krylov space

$$\mathcal{K}_k((s_0\mathbf{I}_N - \mathbf{A})^{-1}, (s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b})$$

  and therefore the knowledge of the action of $(s_0\mathbf{I}_N - \mathbf{A})^{-1} \in \mathbb{R}^{N\times N}$ on vectors; two computationally efficient approaches are possible:

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  - if N is sufficiently small, an LU factorization of $s_0\mathbf{I}_N - \mathbf{A}$ can be performed and for any vector $\mathbf{v} \in \mathbb{R}^N$, $(s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{v}$ can be computed using forward and backward substitutions

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  - if N is sufficiently small, an LU factorization of $s_0\mathbf{I}_N - \mathbf{A}$ can be performed and for any vector $\mathbf{v} \in \mathbb{R}^N$, $(s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{v}$ can be computed using forward and backward substitutions
  - if N is too large for an LU factorization to be affordable, Krylov subspace recycling techniques allowing the reuse of Krylov subspaces for multiple right-hand sides can be used

- $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ can be efficiently constructed using the Arnoldi factorization method

  Input: $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{b} \in \mathbb{R}^N$
  Output: Orthogonal basis $\mathbf{V}_k \in \mathbb{R}^{N \times k}$ for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$

- In this case, $\mathbf{V}_k$ satisfies the following recursion

$$\mathbf{A}\mathbf{V}_k = \mathbf{V}_k \mathbf{H}_k + \mathbf{f}_k \mathbf{e}_k^T$$

  where $\mathbf{H}_k = \mathbf{V}_k^T \mathbf{A} \mathbf{V}_k$ is an upper Hessenberg matrix, $\mathbf{V}_k^T \mathbf{V}_k = \mathbf{I}_k$, and $\mathbf{V}_k^T \mathbf{f}_k = \mathbf{0}$

Algorithm

Input: $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{b} \in \mathbb{R}^N$

Output: Orthogonal basis $\mathbf{V}_k \in \mathbb{R}^{N \times k}$ for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$

$\mathbf{v}_1 = \mathbf{b}/\|\mathbf{b}\|$;

$\mathbf{w} = \mathbf{A}\mathbf{v}_1$; $\alpha_1 = \mathbf{v}_1^T \mathbf{w}$;

$\mathbf{f}_1 = \mathbf{w} - \alpha_1 \mathbf{v}_1$;

$\mathbf{V}_1 = [\mathbf{v}_1]$; $\mathbf{H} = [\alpha_1]$;

for $j = 1, \cdots, k-1$ **do**

  $\beta_j = \|\mathbf{f}_j\|$; $\mathbf{v}_{j+1} = \mathbf{f}_j / \beta_j$;

  $\mathbf{V}_{j+1} = [\mathbf{V}_j, \ \mathbf{v}_{j+1}]$;

  $\hat{\mathbf{H}}_j = \begin{bmatrix} \mathbf{H}_j \\ \beta_j \mathbf{e}_j^T \end{bmatrix}$;

  $\mathbf{w} = \mathbf{A}\mathbf{v}_{j+1}$;

  $\mathbf{h} = \mathbf{V}_{j+1}^T \mathbf{w}$; $\mathbf{f}_{j+1} = \mathbf{w} - \mathbf{V}_{j+1}\mathbf{h}$;

  $\mathbf{H}_{j+1} = [\hat{\mathbf{H}}_j, \ \mathbf{h}]$;

end for

- $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ and $\mathcal{K}_k(\mathbf{A}^T, \mathbf{c}^T)$ can be efficiently simultaneously constructed using the two-sided Lanczos process

  Input: $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{b} \in \mathbb{R}^N$, $\mathbf{c}^T \in \mathbb{R}^N$
  Output: Bi-orthogonal bases $\mathbf{V}_k \in \mathbb{R}^{N \times k}$ and $\mathbf{W}_k \in \mathbb{R}^{N \times k}$ $(\mathbf{W}_k^T \mathbf{V}_k = \mathbf{I}_k)$ for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ and $\mathcal{K}_k(\mathbf{A}^T, \mathbf{c}^T)$, respectively

- In this case, $\mathbf{V}_k$ and $\mathbf{W}_k$ satisfy the following recursions

$$\mathbf{A}\mathbf{V}_k = \mathbf{V}_k \mathbf{T}_k + \mathbf{f}_k \mathbf{e}_k^T$$

$$\mathbf{A}^T \mathbf{W}_k = \mathbf{W}_k \mathbf{T}_k^T + \mathbf{g}_k \mathbf{e}_k^T$$

  where $\mathbf{T}_k = \mathbf{W}_k^T \mathbf{A} \mathbf{V}_k$ is a tridiagonal matrix, $\mathbf{W}_k^T \mathbf{V}_k = \mathbf{I}_k$, $\mathbf{W}_k^T \mathbf{f}_k = \mathbf{0}$, and $\mathbf{V}^T \mathbf{g}_k = \mathbf{0}$

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Input: $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{b} \in \mathbb{R}^{N}$, $\mathbf{c}^T \in \mathbb{R}^{N}$
Output: Bi-orthogonal bases $\mathbf{V}_k \in \mathbb{R}^{N \times k}$ and $\mathbf{W}_k \in \mathbb{R}^{N \times k}$ ($\mathbf{W}_k^T \mathbf{V}_k = \mathbf{I}_k$) for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ and $\mathcal{K}_k(\mathbf{A}^T, \mathbf{c}^T)$, respectively

$\beta_1 = \sqrt{|\mathbf{b}^T \mathbf{c}^T|}$, $\gamma_1 = \text{sign}(\mathbf{b}^T \mathbf{c}^T)\beta_1$;
$\mathbf{v}_1 = \mathbf{b}/\beta_1$, $\mathbf{w}_1 = \mathbf{c}^T/\gamma_1$;
for $j = 1, \cdots, k-1$ **do**
  $\alpha_j = \mathbf{w}_j^T \mathbf{A} \mathbf{v}_j$;
  $\mathbf{r}_j = \mathbf{A}\mathbf{v}_j - \alpha_j \mathbf{v}_j - \gamma_j \mathbf{v}_{j-1}$;
  $\mathbf{q}_j = \mathbf{A}^T \mathbf{w}_j - \alpha_j \mathbf{w}_j - \beta_j \mathbf{w}_{j-1}$;
  $\beta_{j+1} = \sqrt{|\mathbf{r}_j^T \mathbf{q}_j|}$, $\gamma_{j+1} = \text{sign}(\mathbf{r}_j^T \mathbf{q}_j)\beta_{j+1}$;
  $\mathbf{v}_{j+1} = \mathbf{r}_j/\beta_{j+1}$;
  $\mathbf{w}_{j+1} = \mathbf{q}_j/\gamma_{j+1}$;
end for
$\mathbf{V}_k = [\mathbf{v}_1, \cdots, \mathbf{v}_k]$, $\mathbf{W}_k = [\mathbf{w}_1, \cdots, \mathbf{w}_k]$;

Definition

The $\mathcal{H}_2$ norm of a continuous dynamical system $S = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is the $\mathcal{L}_2$ norm of its associated impulse response $\mathbf{h}(\cdot)$.
When $\mathbf{A}$ is stable and $\mathbf{D} = \mathbf{0}$, the norm is bounded and

$$\|S\|_{\mathcal{H}_2} = \left(\int_0^\infty \text{trace}\left(\mathbf{h}^T(t)\mathbf{h}(t)\right) dt\right)^{1/2}$$

- Using Parseval's theorem and the transfer function $\mathbf{H}(\cdot)$, one can obtain the corresponding expression in the frequency domain

$$\|S\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^\infty \text{trace}\left(\mathbf{H}^*(-i\omega)\mathbf{H}(i\omega)\right) d\omega\right)^{1/2}$$

- One can also derive the expression of $\|S\|_{\mathcal{H}_2}$ in terms of the reachability and observability Gramians $\mathcal{P}$ and $\mathcal{Q}$

$$\|S\|_{\mathcal{H}_2} = \sqrt{\text{trace}\left(\mathbf{B}^T\mathcal{Q}\mathbf{B}\right)} = \sqrt{\text{trace}\left(\mathbf{C}\mathcal{P}\mathbf{C}^T\right)}$$

- In the SISO case, the transfer function is a rational function: Assuming (for simplicity) that it has distinct poles λ_i associated with the residues h_i, $i = 1, \cdots, N$ [1], it can be written as

$$\mathbf{H}(s) = \sum_{i=1}^{N} \frac{h_i}{s - \lambda_i}$$

- Then, the following theorem can be established

Theorem

Let $\mathbf{H}_r(\cdot)$ denote the transfer function associated with the reduced system $\mathcal{S}_r$ obtained using moment matching, the Lanczos procedure, and the high-dimensional system $\mathcal{S}$. Denoting by $h_{r,i}$ and $\lambda_{r,i}$, $i = 1, \cdots, k$, the residues and poles of $\mathbf{H}_r(\cdot)$, respectively, the following result holds

$$\|\mathcal{S} - \mathcal{S}_r\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} h_i \left(\mathbf{H}(-\lambda_i^*) - \mathbf{H}_r(-\lambda_i^*)\right) + \sum_{i=1}^{k} h_{r,i} \left(\mathbf{H}_r(-\lambda_{r,i}) - \mathbf{H}(-\lambda_{r,i})\right)$$

[1] When a transfer function is expressed as a sum of simpler fractions, the residues are the coefficients corresponding to each pole

- One would like to build ROBs $(\mathbf{V}, \mathbf{W})$ of a given dimension k such that the corresponding reduced system $\mathcal{S}_r$ is $\mathcal{H}_2$-**optimal**, i.e. solves the following optimization problem

$$\min_{\mathcal{S}_r,\ \text{rank}(\mathbf{V})=\text{rank}(\mathbf{W})=k} \|\mathcal{S} - \mathcal{S}_r\|_{\mathcal{H}_2}$$

- In this case, one can show that a **necessary condition** is that the reduced-order model matches the first two moments of the HDM at the points $-\lambda_{r,i}$, mirror images of the poles $\lambda_{r,i}$ of the reduced transfer function $\mathbf{H}_r(\cdot)$

$$\mathbf{H}_r(-\lambda_{r,i}) = \mathbf{H}(-\lambda_{r,i}), \quad \mathbf{H}_r^{(1)}(-\lambda_{r,i}) = \mathbf{H}^{(1)}(-\lambda_{r,i}), \ s = 1, \cdots, k$$

- Unfortunately, moment matching ensures that the moments of the transfer function are matched at $\lambda_{r,i}$, not $-\lambda_{r,i}$
- The IRKA (Iterative Rational Krylov Approximation) procedure is an iterative procedure to conciliate these two contradicting goals

- POD in the frequency domain for LTI systems

$$\begin{aligned}\text{range}(\mathbf{V}) = &= \text{span}\{\mathcal{X}(\omega_1), \cdots, \mathcal{X}(\omega_k)\} \\ &= \text{span}\left\{(j\omega_1 \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (j\omega_k \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right\}\end{aligned}$$

  where $\omega_1, \cdots, \omega_k \in \mathbb{R}^+$

- Rational interpolation with first moment matching at multiple points

$$\text{range}(\mathbf{V}) = \text{span}\left\{(s_1 \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (s_k \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right\}$$

  where $s_1, \cdots, s_k \in \mathbb{C}$

- Question: is it possible to extend the two-sided moment matching approach to POD?
- Answer: yes, this is the Balanced POD

- The Balanced POD method generates snapshots for the dual system in addition to the POD snapshots

$$\begin{aligned}\mathbf{S} &= \left[(j\omega_1\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b} \;\cdots\; (j\omega_k\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right]\\ \mathbf{S}_{\text{dual}} &= \left[(-j\omega_1\mathbf{I}_N - \mathbf{A}^T)^{-1}\mathbf{c}^T \;\cdots\; (-j\omega_k\mathbf{I}_N - \mathbf{A}^T)^{-1}\mathbf{c}^T\right]\end{aligned}$$

- The associated right and left ROBs are then computed as follows

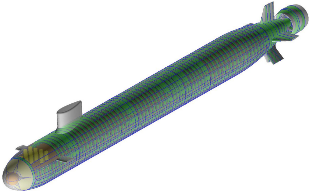
$$\begin{aligned}\mathbf{S}_{\text{dual}}^T\mathbf{S} &= \mathbf{U}\mathbf{\Sigma}\mathbf{Z}^T \quad \text{(SVD)}\\ \mathbf{V} &= \mathbf{S}\mathbf{Z}_k\mathbf{\Sigma}_k^{-1/2}\\ \mathbf{W} &= \mathbf{S}_{\text{dual}}\mathbf{U}_k\mathbf{\Sigma}_k^{-1/2}\end{aligned}$$

  where the subscript k refers to the first k components of the singular value decomposition

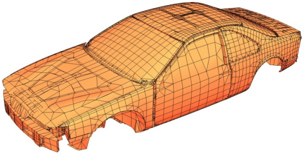
- If no truncation is performed, Balanced POD is equivalent to two-sided moment matching at $s_i \in \{\omega_1, \cdots, \omega_k\}$

- Structural vibrations and interior noise/acoustics

Structural dynamics (Navier)

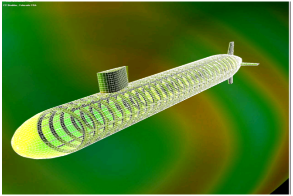


Interior Helmholtz

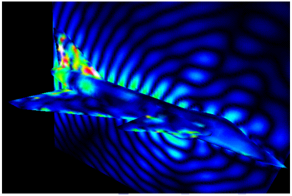


- Scattering (acoustics and electromagnetics)

Exterior Helmholtz



Electromagnetics (Maxwell)
Aeroacoustics (Helmholtz)



- Structural dynamics

$$\mathbf{w}_s(\omega) = \left(\underbrace{\mathbf{K}_s}_{\mathbf{K}} + i\omega \underbrace{\mathbf{D}_s}_{\mathbf{D}} - \omega^2 \underbrace{\mathbf{M}_s}_{\mathbf{M}} \right)^{-1} \mathbf{f}_s(\omega)$$

Rayleigh damping: $\mathbf{D}_s = \alpha \mathbf{K}_s + \beta \mathbf{M}_s$

- Acoustics

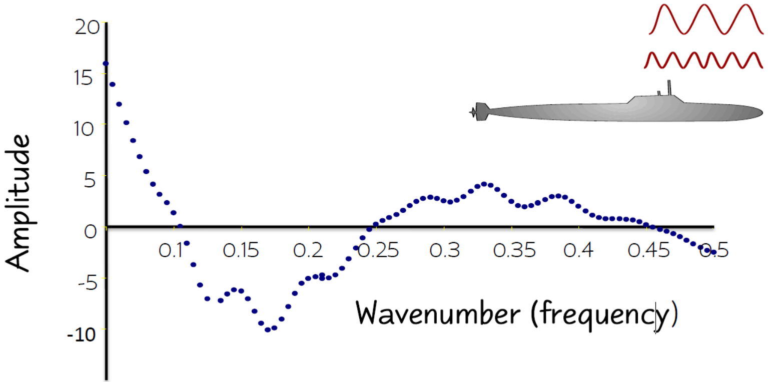
$$\mathbf{w}_f(\omega) = \left(\underbrace{\mathbf{K}_f}_{\mathbf{K}} - \underbrace{\frac{\omega^2}{c_f^2}}_{\omega^2} \underbrace{\mathbf{M}_f}_{\mathbf{M}} + \underbrace{\mathbf{S}_a(\omega)}_{i\omega\mathbf{D}} \right)^{-1} \mathbf{f}_f(\omega)$$

- Structural (or vibro)-acoustics

$$\mathbf{w}_v(\omega) = \left(\mathbf{K}_v - \omega^2 \mathbf{M}_v + \mathbf{S}_v(\omega)\right)^{-1} \mathbf{f}_v(\omega)$$

$$= \left(\underbrace{\begin{bmatrix} \mathbf{K}_s & \mathbf{C}^T \\ \mathbf{0} & \frac{1}{\rho_f}\mathbf{K}_f \end{bmatrix}}_{\mathbf{K}} - \omega^2 \underbrace{\begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ -\mathbf{C} & \frac{1}{\rho_f c_f^2}\mathbf{M}_f \end{bmatrix}}_{\mathbf{M}} + \underbrace{\begin{bmatrix} i\omega\mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{i\omega\mathbf{D}} \right)^{-1} \begin{bmatrix} \mathbf{f}_s(\omega) \\ \frac{1}{\rho_f}\mathbf{f}_f(\omega) \end{bmatrix}$$

- Frequency response function $\mathbf{w} = \mathbf{w}(\omega) \Longrightarrow$ problem with multiple left hand sides - very CPU intensive (1,000s of frequencies)



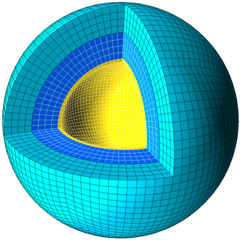
- Approximate $\mathbf{w}(\omega)$ by a Galerkin projection: $\mathbf{w} \approx \tilde{\mathbf{w}} = \mathbf{V}\mathbf{q}$

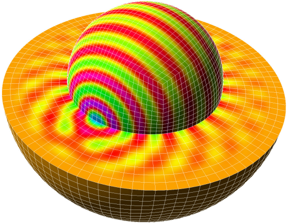
$$\tilde{\mathbf{w}}(\omega) = \mathbf{V} \underbrace{\left(\mathbf{V}^\star\mathbf{K}\mathbf{V} + i\omega\mathbf{V}^\star\mathbf{D}\mathbf{V} - \omega^2\mathbf{V}^\star\mathbf{M}\mathbf{V}\right)^{-1}}_{\textbf{PROM}} \mathbf{V}^\star\mathbf{f}$$

- If the columns of $\mathbf{V}$ span the solution and its derivatives at some frequency, the projection is **interpolatory**
- Two ways to compute the vectors in $\mathbf{V}$
  - recursive differentiation with respect to ω at the interpolating frequency
  - construction of a Krylov space that spans the derivatives (special cases)

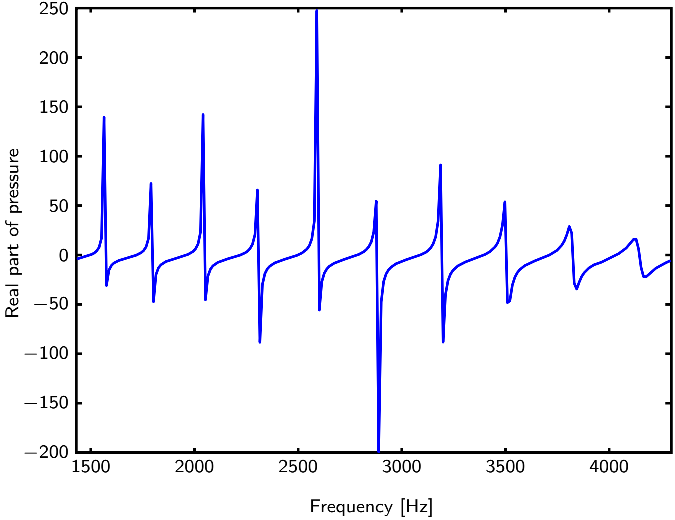
$$\begin{aligned}\mathrm{span}\Big\{&(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{f},\\ &(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{M}(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{f},\\ &\cdots\\ &[(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{M}]^{n-1}(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{f}\Big\}\end{aligned}$$

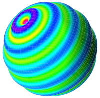
- Frequency sweep analysis of a thick spherical steel shell submerged in water and excited by a point load on its inner surface
- Finite element model using isoparametric cubic elements with roughly $N = 1,200,000$ dofs





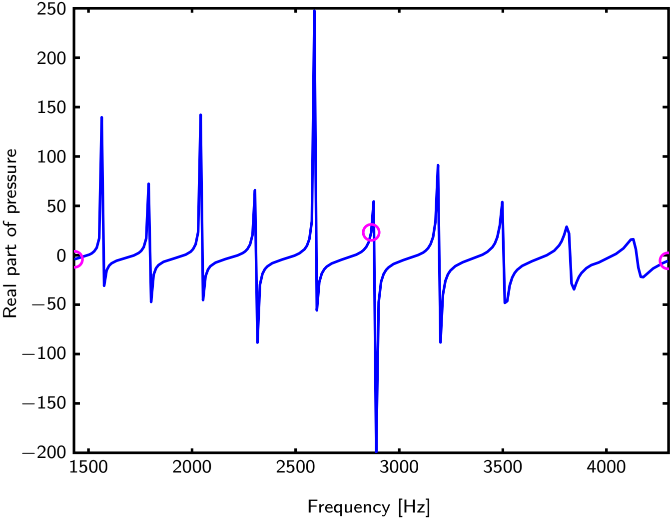
- Frequency sweep analysis of a submerged shell

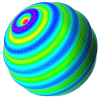




- reference

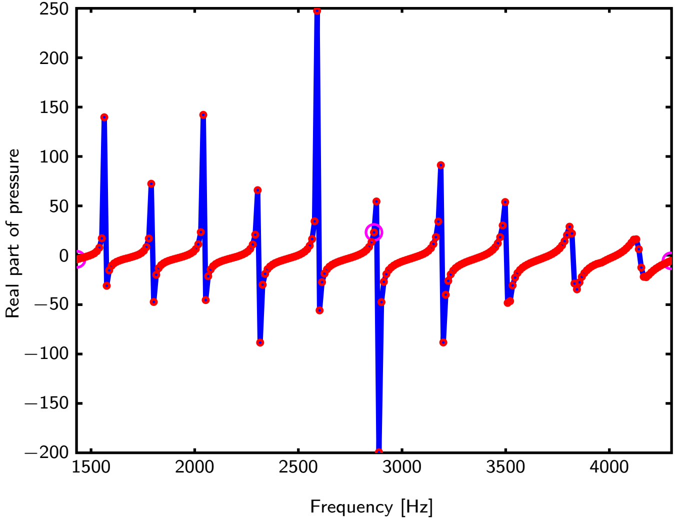
- Frequency sweep analysis of a submerged shell

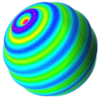




- reference
- interpolating frequencies: 1,430Hz, 2,860Hz, and 4,290Hz

- Frequency sweep analysis of a submerged shell





- reference
- interpolating frequencies: 1,430Hz, 2,860Hz, and 4,290Hz
- 3-points with 32+32+32 vectors
- sampling every 12Hz

- How to choose
  - number of interpolating frequencies
  - location of interpolating frequencies
  - number of derivatives (Krylov vectors)
- Error indicator: relative residual

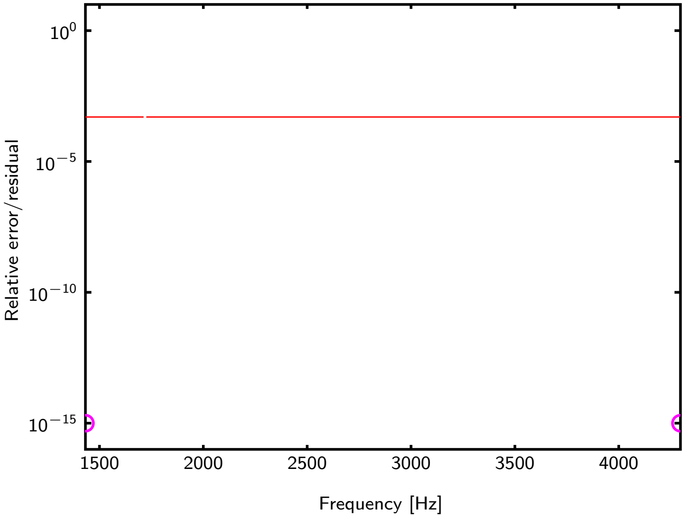
$$\frac{\|(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})\tilde{\mathbf{w}}(\omega) - \mathbf{f}\|}{\|\mathbf{f}\|}$$

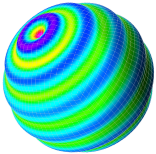
where

$$\tilde{\mathbf{w}}(\omega) = \mathbf{V}\left(\mathbf{V}^\star\mathbf{K}\mathbf{V} + i\omega\mathbf{V}^\star\mathbf{D}\mathbf{V} - \omega^2\mathbf{V}^\star\mathbf{M}\mathbf{V}\right)^{-1}\mathbf{V}^\star\mathbf{f}$$

1. Specify the number of derivatives per frequency and an accuracy threshold
2. Use two interpolations frequencies at the extremities of the frequency band of interest and construct the ROB
3. Evaluate the residual at some *small* set of in between frequencies
4. Add a frequency where the residual is largest and update the projection
5. Repeat until the residual is below a threshold at *all sampling points*
6. Check at the end the residual at all sampled (or user-specified) frequencies

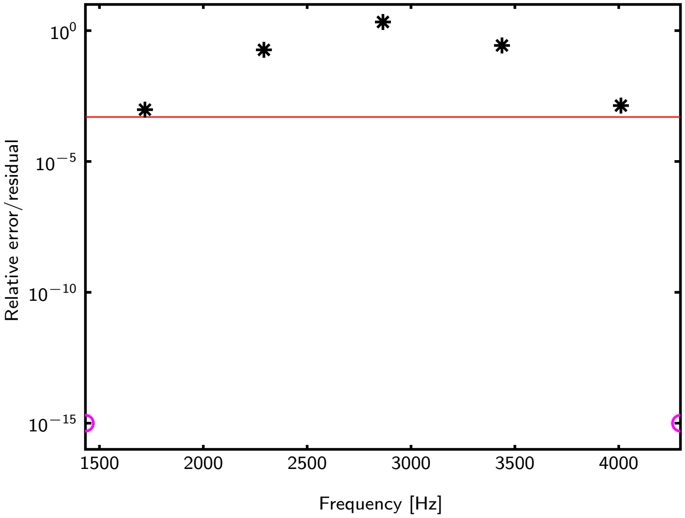
- Frequency sweep analysis of a submerged shell

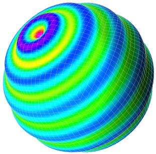




- PROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers

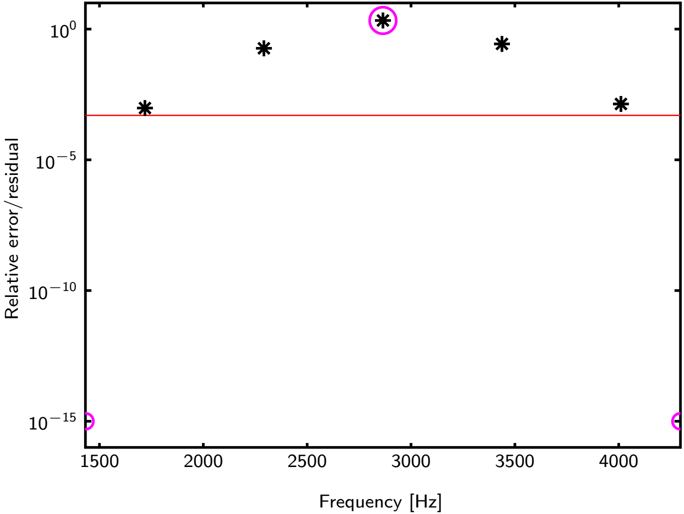
- Frequency sweep analysis of a submerged shell

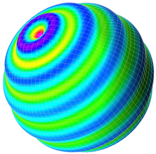




- PROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *

- Frequency sweep analysis of a submerged shell





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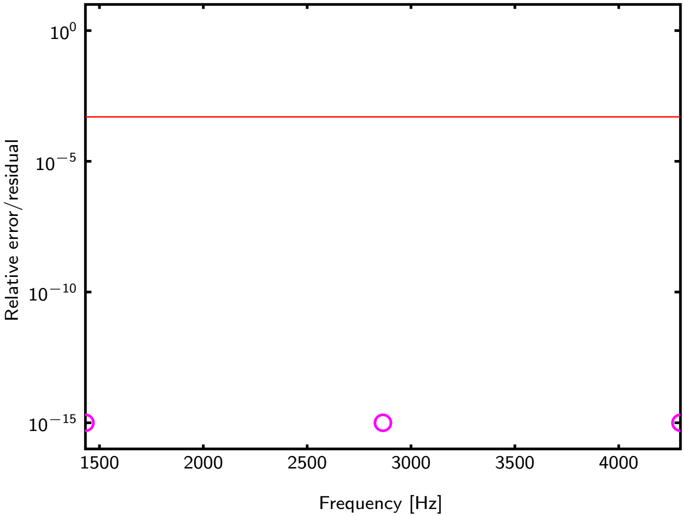

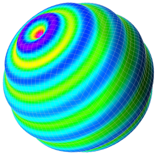

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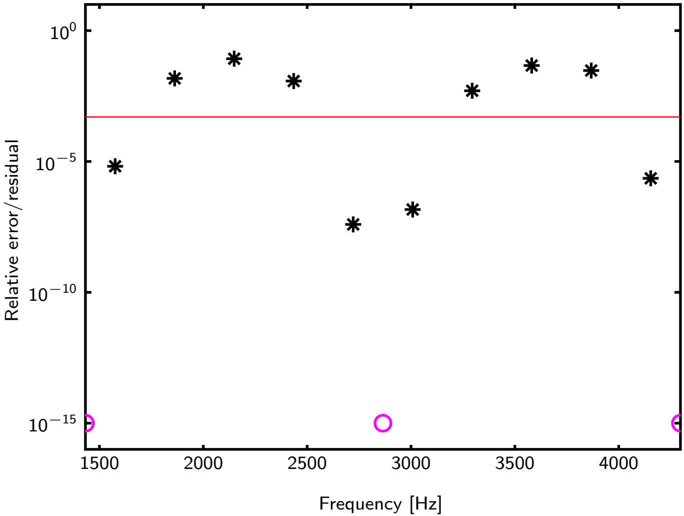

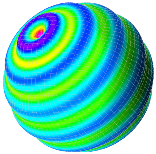

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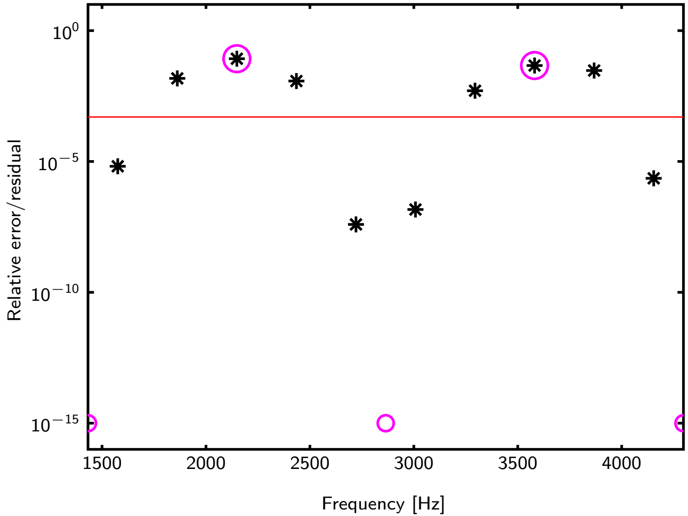
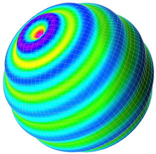
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *

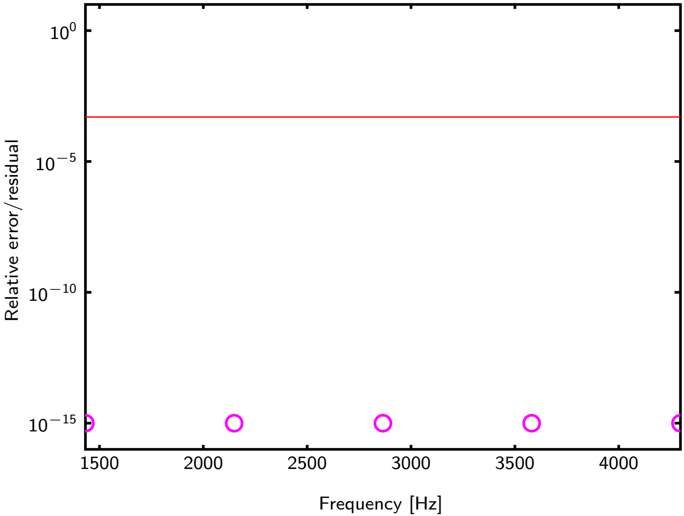
- Frequency sweep analysis of a submerged shell

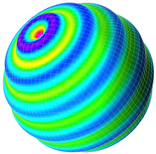




- PROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *

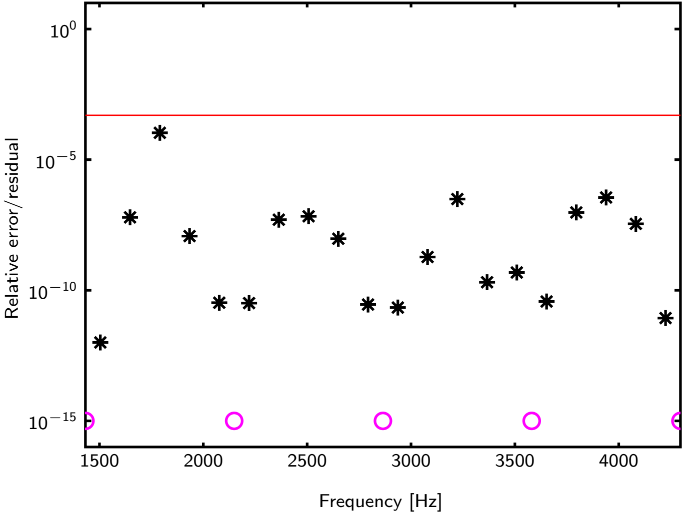
- Frequency sweep analysis of a submerged shell

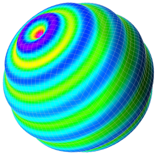




- PROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *

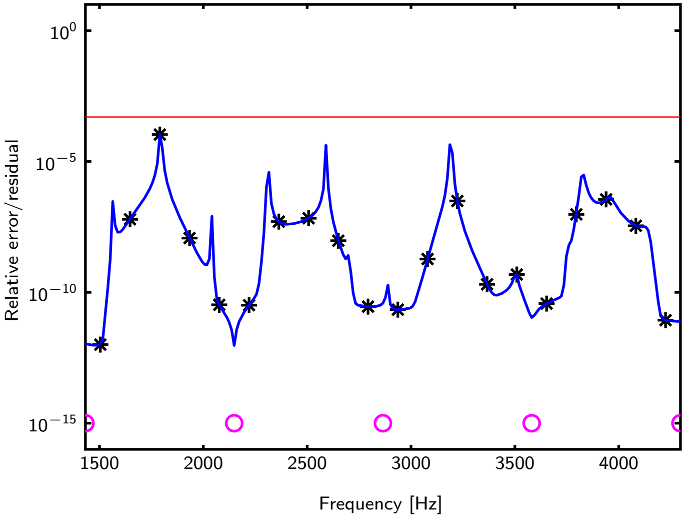
- Frequency sweep analysis of a submerged shell

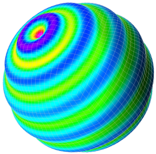




- PROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance $=$ 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *

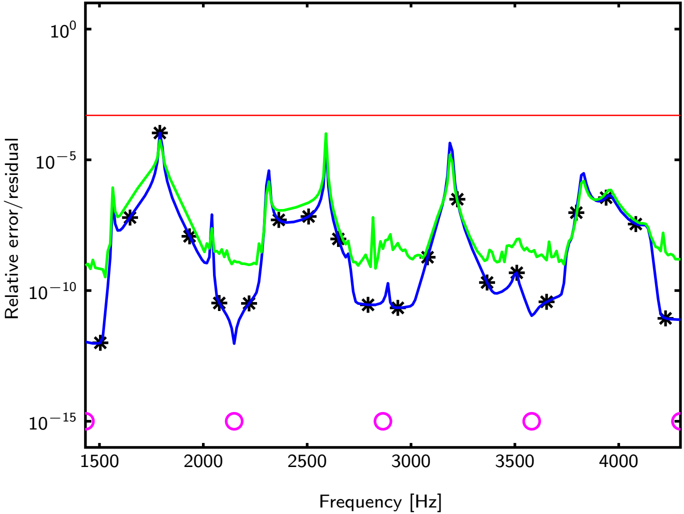
- Frequency sweep analysis of a submerged shell

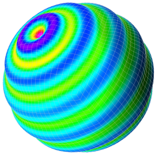




- PROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *
- final residual

- Frequency sweep analysis of a submerged shell





- PROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance $=$ 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *
- final residual
- final error