AA216/CME345: PROJECTION-BASED MODEL ORDER REDUCTION

Linear Dynamical Systems

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Outline

- Input function of interest

$$\begin{aligned} \mathbf{u} \quad : \quad & \mathbb{R} \rightarrow \mathbb{U} \subset \mathbb{R}^{in} \\ & t \longmapsto \mathbf{u}(t) \end{aligned}$$

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- *Assumption:* There exists a linear operator $\mathcal{S}$ that maps the input space $\mathbb{U}$ to the output space $\mathbb{Y}$

$$\begin{aligned} \mathcal{S} \quad : \quad & \mathbb{U} \rightarrow \mathbb{Y} \\ & \mathbf{u} \longmapsto \mathbf{y}(\mathbf{u}) \end{aligned}$$

- A system considered here can be characterized by

$$\mathcal{S} : \mathbf{u} \longmapsto \mathbf{y}, \;\; \mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(t,\tau)\mathbf{u}(\tau)d\tau$$

where $\mathbf{h}(t,\tau) \in \mathbb{R}^{q \times in}$, called the kernel of the system, represents the system's impulse response and describes how the system reacts over time to an impulse applied at τ

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  - time-invariance: applying an input to a system considered here now or t_0 seconds later will lead to identical outputs except for a time delay of t_0 seconds $\Rightarrow$ the output is independent of the specific time at which the input is applied

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  - theorem: for a time-invariant lineary dynamical system, the following property holds

$$\forall(t,\tau), \quad \mathbf{h}(t,\tau) = \mathbf{h}(t-\tau)$$

- Proof
  - the output for an input shifted by t_0 is

$$\begin{aligned}\mathbf{y}^{\text{shf}}(t) &= \int_{-\infty}^{+\infty} \mathbf{h}(t,\tau)\mathbf{u}(\tau+t_0)d\tau = \int_{-\infty}^{+\infty} \mathbf{h}\left(t,\tau'-t_0\right)\mathbf{u}(\tau')d\tau' \\ &= \int_{-\infty}^{+\infty} \mathbf{h}(t,\tau-t_0)\mathbf{u}(\tau)d\tau\end{aligned}$$

  - the delayed output for the original input is
$\mathbf{y}^{\text{dly}}(t+t_0) = \int_{-\infty}^{+\infty} \mathbf{h}(t+t_0,\tau)\mathbf{u}(\tau)d\tau$
  - time-invariance implies that

$$\mathbf{y}^{\text{shf}}(t) = \mathbf{y}^{\text{dly}}(t+t_0) \Rightarrow \int_{-\infty}^{+\infty} \left(\mathbf{h}(t,\tau-t_0) - \mathbf{h}(t+t_0,\tau)\right)\mathbf{u}(\tau)d\tau = \mathbf{0}$$

  - for the above result to hold $\forall t,\ t_0$, and $\mathbf{u}$, the kernel $\mathbf{h}$ must satisfy

$$\forall t,\tau,t_0 \;\; \mathbf{h}(t,\tau-t_0) = \mathbf{h}(t+t_0,\tau) \Rightarrow \forall (t,\tau), \;\; \mathbf{h}(t,\tau) = \mathbf{h}(t-\tau) \;\; \square$$

- Consequences
  - $\mathcal{S} : \mathbf{u} \longmapsto \mathbf{y}, \;\; \mathbf{y}(t) = \int_{-\infty}^{+\infty} \mathbf{h}(t-\tau)\mathbf{u}(\tau)d\tau$

    $\Longrightarrow \mathcal{S}$ is called a convolution operator

    $$\boxed{\mathbf{y} = \mathcal{S}(\mathbf{u}) = \mathbf{h} * \mathbf{u}}$$

  - "commutativity"

    $$\begin{aligned}\mathbf{h} * \mathbf{u} = & \int_{-\infty}^{+\infty} \mathbf{h}(t-\tau)\mathbf{u}(\tau)d\tau = -\int_{+\infty}^{-\infty} \mathbf{h}(\tau')\mathbf{u}(t-\tau')d\tau' \\ = & \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{u}(t-\tau)d\tau \left(= \int_{-\infty}^{+\infty} \mathbf{u}(t-\tau)\mathbf{h}(\tau)d\tau = \mathbf{u} * \mathbf{h}, \text{ if } q = in \right.\end{aligned}$$

  - time delay on input $\mathbf{u}(t+t_0)$ equates to time delay of output $\mathbf{y}(t+t_0)$

    $$\begin{aligned}\int_{-\infty}^{+\infty} \mathbf{h}(t-\tau)\mathbf{u}(\tau+t_0)d\tau &= -\int_{+\infty}^{-\infty} \mathbf{h}(\tau')\mathbf{u}(t+t_0-\tau')d\tau' \\ &= \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{u}(t+t_0-\tau)d\tau = \mathbf{y}(t+t_0)\end{aligned}$$

- Further assumption
  - causality: the output of a system considered here depends only on present and past inputs

$$\forall \tau > t, \quad \mathbf{h}(t,\tau) = 0 \quad \Rightarrow \quad \mathbf{y}(t) = \int_{-\infty}^{t} \mathbf{h}(t,\tau)\mathbf{u}(\tau)d\tau$$

- Consequence

$$\boxed{\mathcal{S} : \mathbf{u} \longmapsto \mathbf{y}, \ \ \mathbf{y}(t) = \int_{-\infty}^{t} \mathbf{h}(t-\tau)\mathbf{u}(\tau)d\tau}$$

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- Theorem: There are two components in the output

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- Theorem: There are two components in the output
  - instantaneous component, which depends directly on the current input value (in the impulse response framework, this is captured by the value of the impulse response at $\tau = t$ – that is, $\mathbf{h}(0)$)

$$\mathbf{h}_0\mathbf{u}(t) \ \text{ where } \ \mathbf{h}_0 = \mathbf{h}(0) \in \mathbb{R}^{in \times q}$$

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  - dynamic component resulting from past or future input influences and capturing the system's dynamics such as memory or feedback (in the integral formulation, these influences occur when $\tau \neq t$)

    $$\int_{-\infty}^{t} \mathbf{h}_d(t-\tau)\mathbf{u}(\tau)d\tau \;\; \text{where} \;\; \mathbf{h}_d \text{ is a smooth function}$$

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  - kernel function: $\mathbf{h}(t) = \mathbf{h}_0\delta(t) + \mathbf{h}_d(t), \ \ \forall t \geq 0$
    where δ is the Dirac delta function and thus $\mathbf{h}(t)$ is the response of the system to an impulse δ (impulse response)

- Proof
  - decompose $\mathbf{h}(t)$ as $\mathbf{h}(t) = \mathbf{h}_0\delta(t) + \mathbf{h}_d(t)$
  - substitute in the expression of the output to obtain

$$
\begin{aligned}
\mathbf{y}(t) &= \int_{-\infty}^{t} \mathbf{h}(t-\tau)\mathbf{u}(\tau)d\tau \\
&= \int_{-\infty}^{t} \mathbf{h}_0\delta(t-\tau)\mathbf{u}(\tau)d\tau \;+\; \int_{-\infty}^{t} \mathbf{h}_d(t-\tau)\mathbf{u}(\tau)d\tau \\
&= \mathbf{h}_0\mathbf{u}(t) \;+\; \underbrace{\int_{-\infty}^{t} \mathbf{h}_d(t-\tau)\mathbf{u}(\tau)d\tau}_{\substack{\text{accounts for the}\\ \text{accumulated effect of}\\ \text{the input over time due}\\ \text{to the system's dynamics}}}
\end{aligned}
$$

- Laplace transform (time-domain $\rightarrow$ Laplace s-domain)

$$\boxed{F(s) = \mathcal{L}\left(f(t)\right) = \int_0^{\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}}$$

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$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) ds, \quad \gamma \in \mathbb{R}$$

where γ is such that the contour path of integration is in the region of convergence of $F(s)$

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- The inverse Laplace transform is also known as the Bromwich integral, the Fourier-Mellin integral, or Mellin's inverse formula

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- Input-output mapping

$$\mathcal{L}(\mathbf{y}) = \mathcal{L}(\mathbf{h} * \mathbf{u}) \Rightarrow \boxed{\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)}$$

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- $\mathbf{H}(s)$ is known as a transfer function

- A time-continuous linear dynamical system has the form

$$\begin{aligned} \frac{d\mathbf{w}}{dt}(t) &= \mathbf{A}\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{w}(t) + \mathbf{D}\mathbf{u}(t) \\ \mathbf{w}(t_0) &= \mathbf{w}_0 \end{aligned}$$

  - $t \in [t_0, \infty)$
  - $\mathbf{w} \in \mathbb{W} \subset \mathbb{R}^N$: Vector of state variables belonging to state domain $\mathbb{W}$
  - $\mathbf{u} \in \mathbb{U} \subset \mathbb{R}^{in}$: Vector of input variables — typically $in \ll N$
  - $\mathbf{y} \in \mathbb{Y} \subset \mathbb{R}^q$: Vector of output variables — typically $q \ll N$
  - $\mathbf{A} \in \mathbb{R}^{N \times N}$: Dynamical operator
  - $\mathbf{B} \in \mathbb{R}^{N \times in}$: Input operator
  - $\mathbf{C} \in \mathbb{R}^{q \times N}$ and $\mathbf{D} \in \mathbb{R}^{q \times in}$: Output operators

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$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$$

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- Hence, a time-continuous linear dynamical system can be represented by the dynamical quadruplet

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- In the following, $\mathbb{W} = \mathbb{R}^N$, $\mathbb{U} = \mathbb{R}^{in}$ and $\mathbb{Y} = \mathbb{R}^q$

- The solution $\mathbf{w}(t)$ of the above linear ODE is the function $\phi(t, \mathbf{u}; t_0, \mathbf{w}_0)$ given by

$$\boxed{\phi(t, \mathbf{u}; t_0, \mathbf{w}_0) = \underbrace{e^{\mathbf{A}(t-t_0)}\mathbf{w}(t_0)}_{homogeneous} + \underbrace{\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau,}_{particular} \ \forall t \geq t_0}$$

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- The corresponding output is (by linearity)

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{C}\phi(t, \mathbf{u}; t_0, \mathbf{w}_0) + \mathbf{D}\mathbf{u}(t) \\
&= \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{w}(0) + \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \\
&= \mathbf{C}\phi(t, \mathbf{0}; t_0, \mathbf{w}_0) + \mathbf{C}\phi(t, \mathbf{u}; t_0, \mathbf{0}) + \mathbf{D}\mathbf{u}(t)
\end{aligned}$$

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&= \mathbf{C}\phi(t, \mathbf{0}; -\infty, \mathbf{0}) + \mathbf{C}\phi(t, \mathbf{u}; -\infty, \mathbf{0}) + \mathbf{D}\mathbf{u}(t) \\
&= \mathbf{0} + \mathbf{C}\phi(t, \mathbf{u}; -\infty, \mathbf{0}) + \mathbf{D}\mathbf{u}(t) \\
&= \int_{-\infty}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \\
&= \int_{-\infty}^{t} \mathbf{h}_d(t-\tau)\mathbf{u}(\tau)d\tau + \mathbf{h}_0\mathbf{u}(t)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{h}_d(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{B} \quad \text{if } t \geq 0 \\
\mathbf{h}_0 &= \mathbf{D}
\end{aligned}$$

- Kernel function (revisited)

$$\mathbf{h}(t) = \mathbf{h}_d(t) + \delta(t)\mathbf{h}_0 = \begin{cases} \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \delta(t)\mathbf{D} & t \geq 0 \\ \mathbf{0} & t < 0 \end{cases}$$

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- Transfer function

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}, \quad s \in \mathbb{C}$$

- Consider the change of variables of the form

$$\tilde{\mathbf{w}} = \mathbf{T}\mathbf{w} \in \mathbb{R}^N$$

where $\mathbf{T} \in \mathbb{R}^{N \times N}$ is nonsingular $\left(\text{i.e. } \mathbf{T} \in \mathrm{GL}(N)\right)$

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- And consider the transformed governing linear ODE

$$\frac{d\tilde{\mathbf{w}}}{dt} = \mathbf{T}\frac{d\mathbf{w}}{dt} = \mathbf{T}\mathbf{A}\mathbf{w} + \mathbf{T}\mathbf{B}\mathbf{u} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\tilde{\mathbf{w}} + \mathbf{T}\mathbf{B}\mathbf{u}$$

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- The corresponding transformed output equation is

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- And thus the transformed dynamical quadruplet is

$$(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{C}}, \widetilde{\mathbf{D}}) = (\mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \mathbf{T}\mathbf{B}, \mathbf{C}\mathbf{T}^{-1}, \mathbf{D})$$

- Particular case: Orthogonal change of variables

$$\tilde{\mathbf{w}} = \mathbf{Q}\mathbf{w} \in \mathbb{R}^N$$

where $\mathbf{Q} \in \mathbb{R}^{N \times N}$ is orthogonal

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}_N \Rightarrow \mathbf{Q}^{-1} = \mathbf{Q}^T$$

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- Norm preservation property

$$\|\tilde{\mathbf{w}}\|_2 = \|\mathbf{Q}\mathbf{w}\|_2 = \sqrt{\mathbf{w}^T\mathbf{Q}^T\mathbf{Q}\mathbf{w}} = \sqrt{\mathbf{w}^T\mathbf{w}} = \|\mathbf{w}\|_2$$

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- In this case, the transformed dynamical quadruplet is

$$(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{C}}, \widetilde{\mathbf{D}}) = (\mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{Q}\mathbf{B}, \mathbf{C}\mathbf{Q}^T, \mathbf{D})$$

Definition

The time-continuous linear system defined by the quadruplet $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is stable if all of the eigenvalues of $\mathbf{A}$ have negative real parts

$$\left(\text{recall } \phi(t, \mathbf{u}; t_0, \mathbf{w}_0) = \underbrace{e^{\mathbf{A}(t-t_0)}\mathbf{w}(t_0)}_{homogeneous} + \underbrace{\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau, \ \forall t \geq t_0}_{particular} \right)$$

Example

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, \ \mathbf{D} = [1]$$

$\lambda(\mathbf{A}) = \{-2, -5\}$

- Response to unit step input $\mathbf{u}(t) = \mathbf{1}_{t\in[0,1]}$