AA216/CME345: PROJECTION-BASED MODEL ORDER REDUCTION

Parameterized Partial Differential Equations (PDEs)

Charbel Farhat
Stanford University
cfarhat@stanford.edu

Outline

- Linear or Nonlinear Partial Differential Equation (PDE)

$$\mathcal{L}(\mathcal{W}, \mathbf{x}, t) = 0$$

  - $\mathcal{W} = \mathcal{W}(\mathbf{x}, t) \in \mathbb{R}^{\ell}$: State variable
  - $\mathbf{x} \in \Omega \subset \mathbb{R}^{d}$, $d \leq 3$: Space variable
  - $t \geq 0$: Time variable
- Examples
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  - elastodynamic equations of motion
  - wave equation

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- Initial Condition (IC)

$$\mathcal{W}(\mathbf{x}, 0) = \mathcal{W}_0(\mathbf{x}) = \mathcal{W}_{\mathrm{IC}}(\mathbf{x})$$

- Parameter domain: $\mathcal{D} \subset \mathbb{R}^p$
  - parameter vector (also referred to as parameter "point"): $\boldsymbol{\mu} = [\mu_1 \; \cdots \; \mu_p]^T \in \mathcal{D} \subset \mathbb{R}^p$

  where the superscript T designates the transpose operation

- Parameterized PDE

$$\mathcal{L}(\mathcal{W}, \mathbf{x}, t; \boldsymbol{\mu}) = 0$$

- Boundary conditions

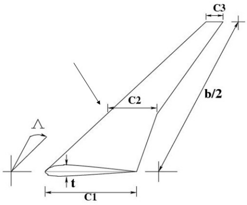
$$\mathcal{B}(\mathcal{W}, \mathbf{x}_{\mathrm{BC}}, t; \boldsymbol{\mu}) = 0$$

- Initial condition

$$\mathcal{W}_0(\mathbf{x}) = \mathcal{W}_{\mathrm{IC}}(\mathbf{x}; \boldsymbol{\mu})$$

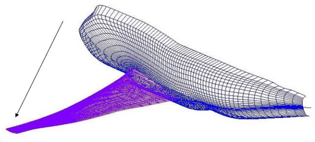
- Physical parameters
  - shape parameters
  - material (properties) parameters
  - operation parameters (for example, flight conditions, cruise conditions, $\cdots$)
  - boundary conditions
  - initial condition

6 planform variables
(sweep, span, t/c, 3 chords)



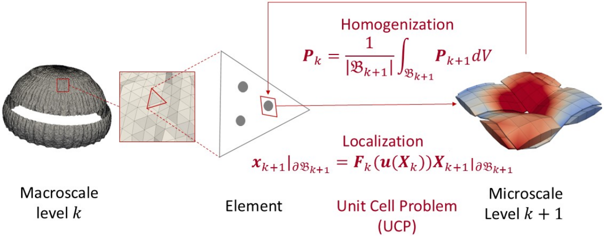
Simplified Planform Model

+ 4 224 mesh points on the wing
surface as design variables



CAD-Free Section Definition

- Other types of parameters
  - modeling parameters
  - abstract parameters



Input to the UCP: 9 components of the deformation gradient F_k
Output of the UCP: 3 components of the symmetric plane stress tensor

→ Parameterization of the UCP: 9 components of F_k

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  - a finite difference method
  - a finite volume method
  - a finite element method
  - a discontinuous Galerkin method
  - a spectral method ...

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$$\boxed{\frac{d\mathbf{w}}{dt} = \mathbf{f}(\mathbf{w}, t; \boldsymbol{\mu})}$$

where

$$\mathbf{w} = \mathbf{w}(t; \boldsymbol{\mu}) \in \mathbb{R}^N$$

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- This is the High-Dimensional Model (HDM)

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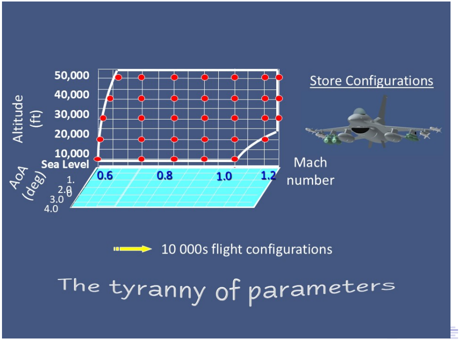
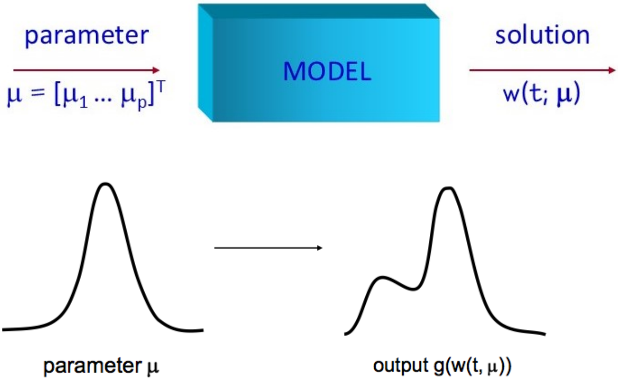
  - model predictive control

- Routine analysis

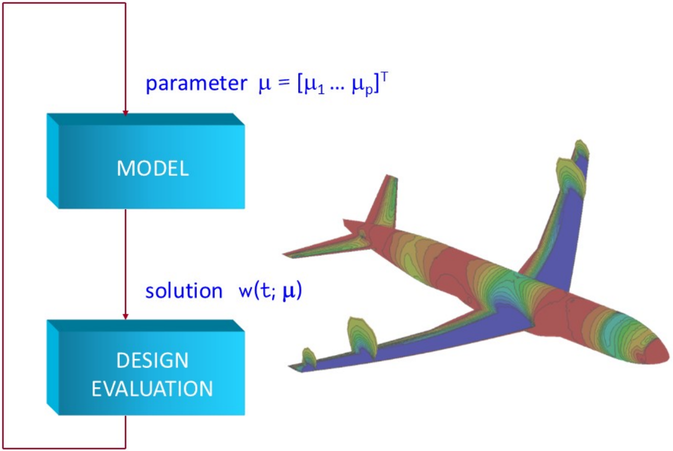


- Uncertainty quantification

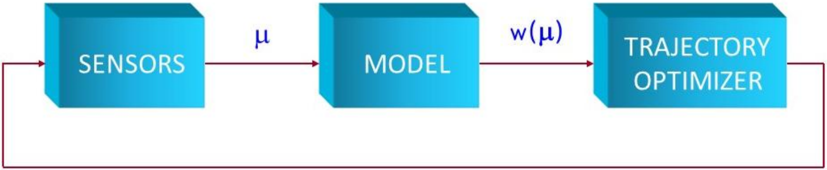


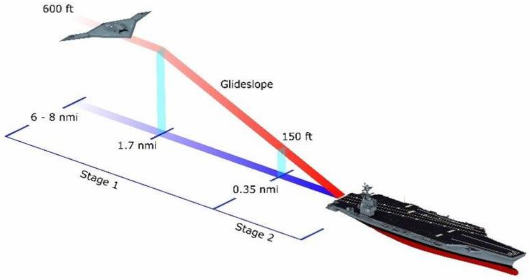
Monte-Carlo simulations ...

- Design optimization



- Model predictive control





- Advection-diffusion-reaction equation: $\mathcal{W} = \mathcal{W}(\mathbf{x}, t; \boldsymbol{\mu})$ solution of

$$\frac{\partial \mathcal{W}}{\partial t} + \mathcal{U} \cdot \nabla \mathcal{W} - \kappa \Delta \mathcal{W} = f_{\mathrm{R}}(\mathcal{W}, t, \boldsymbol{\mu}_{\mathrm{R}}) \text{ for } \mathbf{x} \in \Omega$$

with appropriate boundary and initial conditions

$$\mathcal{W}(\mathbf{x}, t; \boldsymbol{\mu}) = \mathcal{W}_D(\mathbf{x}, t; \boldsymbol{\mu}_{\mathrm{D}}) \text{ for } \mathbf{x} \in \Gamma_{\mathrm{D}}$$

$$\nabla \mathcal{W}(\mathbf{x}, t; \boldsymbol{\mu}) \cdot \mathbf{n}(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \Gamma_{\mathrm{N}}$$

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- Parameters of interest

$$\boldsymbol{\mu} = \left[\mathcal{U}_1 \ \cdots \ \mathcal{U}_d \ \kappa \ \boldsymbol{\mu}_{\mathrm{R}} \ \boldsymbol{\mu}_{\mathrm{D}} \ \boldsymbol{\mu}_{\mathrm{IC}}\right]^T$$

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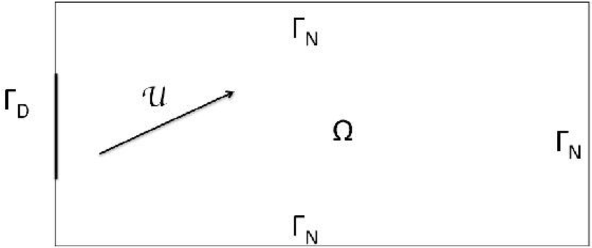

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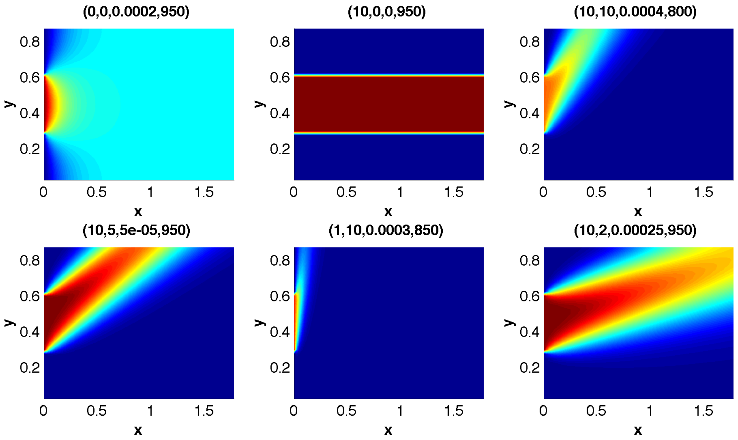
- 4 parameters of interest $\Rightarrow \ p = 4$

$$\boldsymbol{\mu} = [\mathcal{U}_1 \ \mathcal{U}_2 \ \kappa \ \boldsymbol{\mu}_{\mathrm{D}}]^T \in \mathbb{R}^4$$

  where $\boldsymbol{\mu}_{\mathrm{D}}$ is a specified constant value of $\mathcal{W}_D(\mathbf{x}, t; \boldsymbol{\mu}_{\mathrm{D}})$
- $\mathbf{w} \in \mathbb{R}^N$ with $N = 5\,402$

- Solution snapshots at some time t_i, for six sampled parameter points $\boldsymbol{\mu}^{(j)}$, $j = 1, \cdots, 6$ (recall that $\boldsymbol{\mu} = [\mathcal{U}_1\ \mathcal{U}_2\ \kappa\ \boldsymbol{\mu}_{\rm D}]^T \in \mathbb{R}^4$)



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- Idea: Use a linear combination of these snapshots such as, for example

$$\mathbf{w}(t; \boldsymbol{\mu}^\star) \approx \sum_{i=1}^{N_s^{(1)}} q_i^{(1)}(t; \boldsymbol{\mu}^\star)\mathbf{w}(t_i; \boldsymbol{\mu}^{(1)}) + \cdots + \sum_{i=1}^{N_s^{(k)}} q_i^{(k)}(t; \boldsymbol{\mu}^\star)\mathbf{w}(t_i; \boldsymbol{\mu}^{(k)})$$

where

- $N_s^{(j)}$, $j = 1, \cdots, k$ denotes the number of pre-computed solution snapshots using the sampled parameter point $\boldsymbol{\mu}^{(j)}$ and k denotes the total number of parameter points sampled in the parameter space $\mathcal{D}$
- $\mathbf{w}(t_i; \boldsymbol{\mu}^{(j)}) \in \mathbb{R}^N$ denotes the pre-computed solution snapshots at time t_i using the sampled parameter point $\boldsymbol{\mu}^{(j)}$
- $q_i^{(j)}(t; \boldsymbol{\mu}) \in \mathbb{R}$ denotes the expansion coefficient associated with $\mathbf{w}(t_i; \boldsymbol{\mu}^{(j)})$

- The linear expansion

$$\mathbf{w}(t;\boldsymbol{\mu}) \approx \sum_{i=1}^{N_s^{(1)}} q_i^{(1)}(t;\boldsymbol{\mu})\mathbf{w}(t_i;\boldsymbol{\mu}^{(1)}) + \cdots + \sum_{i=1}^{N_s^{(k)}} q_i^{(k)}(t;\boldsymbol{\mu})\mathbf{w}(t_i;\boldsymbol{\mu}^{(k)})$$

can be written as

$$\mathbf{w}(t;\boldsymbol{\mu}) \approx \mathbf{W}\mathbf{q}(t;\boldsymbol{\mu})$$

where

$$\mathbf{W} = \left[\mathbf{w}(t_1;\boldsymbol{\mu}^{(1)}) \ \cdots \ \mathbf{w}(t_{N_s^{(1)}};\boldsymbol{\mu}^{(1)}) \ \cdots \ \mathbf{w}(t_1;\boldsymbol{\mu}^{(k)}) \ \cdots \ \mathbf{w}(t_{N_s^{(k)}};\boldsymbol{\mu}^{(k)})\right]$$

and

$$\mathbf{q}(t;\boldsymbol{\mu}) = \left[q_1^{(1)}(t;\boldsymbol{\mu}) \ \cdots \ q_{N_s^{(1)}}^{(1)}(t;\boldsymbol{\mu}) \ \cdots \ q_1^{(k)}(t;\boldsymbol{\mu}) \ \cdots \ q_{N_s^{(k)}}^{(k)}(t;\boldsymbol{\mu})\right]^T$$

- The parameterized approximation

$$\mathbf{w}(t;\boldsymbol{\mu}) \approx \mathbf{W}\mathbf{q}(t;\boldsymbol{\mu})$$

is a *subspace approximation* of $\mathbf{w}(t;\boldsymbol{\mu})$, where the subspace is

$$\mathcal{S} = \text{span}\left\{\mathbf{w}(t_1;\boldsymbol{\mu}^{(1)}), \cdots, \cdots, \mathbf{w}(t_{N_s^{(k)}};\boldsymbol{\mu}^{(k)})\right\}$$

and its dimension is

$$\dim(\mathcal{S}) = \text{rank}\left[\mathbf{w}(t_1;\boldsymbol{\mu}^{(1)}) \ \cdots \ \cdots \ \mathbf{w}(t_{N_s^{(k)}};\boldsymbol{\mu}^{(k)})\right] \leq \sum_{j=1}^{k} N_s^{(j)}$$

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- This approximation constitutes one of the pillars of projection-based model order reduction (PMOR): It raises the following questions
  - how to sample the parameter space $\mathcal{D}$?
  - how to reduce the dimensionality of $\mathbf{W}$ and therefore that of the approximation subspace $\mathcal{S}$ below $\sum_{j=1}^{k} N_s^{(j)}$?
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- These are *some* of the questions that this course addresses

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- Exponential growth of N_p with p and linear growth of the training cost with N_p $\Rightarrow$ *adaptive* sampling and additional strategies for mitigating the curse of dimensionality