Stabilized Optimization Via an NCL Algorithm

Ding Ma, Kenneth L. Judd, Dominique Orban and Michael A. Saunders

Abstract For optimization problems involving many nonlinear inequality constraints, we extend the bound-constrained (BCL) and linearly constrained (LCL) augmented Lagrangian approaches of LANCELOT and MINOS to an algorithm that solves a sequence of nonlinearly constrained augmented Lagrangian subproblems whose nonlinear constraints satisfy the LICQ everywhere. The NCL algorithm is implemented in AMPL and tested on large instances of a tax policy model that could not be solved directly by the state-of-the-art solvers that we tested, because of singularity in the Jacobian of the active constraints. Algorithm NCL with IPOPT as subproblem solver proves to be effective, with IPOPT using second derivatives and successfully warm starting each subproblem.

Keywords Stabilized optimization · LICQ · Augmented Lagrangian · BCL NCL · Interior method · Warm start

D. Ma
Management Science and Engineering,
Stanford University, Stanford, CA, USA
e-mail: dingma@stanford.edu

K. L. Judd
Hoover Institution, Stanford University, Stanford, CA, USA
e-mail: judd@stanford.edu

D. Orban
GERAD and Department of Mathematics and Industrial Engineering,
École Polytechnique, Montréal, QC, Canada
e-mail: dominique.orban@gerad.ca

M. A. Saunders (✉)
Department of Management Science and Engineering,
Stanford University, Stanford, CA, USA
e-mail: saunders@stanford.edu

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1 Introduction

We consider constrained optimization problems of the form

$$\boxed{\begin{array}{lll}\text{NCO} \qquad & \underset{x\in\mathbb{R}^n}{\text{minimize}} & \phi(x) \\ & \text{subject to} & c(x) \ge 0, \quad Ax \ge b, \quad \ell \le x \le u,\end{array}}$$

where $\phi(x)$ is a smooth nonlinear function, $c(x) \in \mathbb{R}^m$ is a vector of smooth nonlinear functions, and $Ax \ge b$ is a placeholder for a set of linear inequality or equality constraints, with x lying between lower and upper bounds ℓ and u.

In some applications where $m \gg n$, there may be more than n constraints that are essentially active at a solution. The constraints do not satisfy the linear independence constraint qualification (LICQ), and general-purpose solvers are likely to have difficulty converging. Some form of regularization is required. The stabilized SQP methods of Wright [20] and Gill et al. [9, 10] have been developed specifically for such problems. We achieve reliability more simply by adapting the augmented Lagrangian algorithm of the general-purpose optimization solver LANCELOT [4, 5, 15] in the vein of Arreckx and Orban [2] to derive a sequence of regularized subproblems denoted in the next section by NC$_k$.

2 BCL, LCL, and NCL Methods

The theory for the large-scale solver LANCELOT is best described in terms of the general optimization problem

$$\boxed{\begin{array}{lll}\text{NECB} \qquad & \underset{x\in\mathbb{R}^n}{\text{minimize}} & \phi(x) \\ & \text{subject to} & c(x) = 0, \quad \ell \le x \le u\end{array}}$$

with *nonlinear equality constraints* and bounds. We let x^* denote a local solution of NECB and (y^*, z^*) denote associated multipliers. LANCELOT treats NECB by solving a sequence of *bound-constrained subproblems* of the form

$$\boxed{\begin{array}{lll}\text{BC}_k \qquad & \underset{x}{\text{minimize}} & L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2 \\ & \text{subject to} & \ell \le x \le u,\end{array}}$$

where y_k is an estimate of the Lagrange multipliers y^* for the equality constraints. This was called a bound-constrained Lagrangian (BCL) method by Friedlander and Saunders [8], in contrast to the linearly constrained Lagrangian methods (LCL) of Robinson [18] and MINOS [16], whose subproblems LC$_k$ contain bounds as in BC$_k$

and also linearizations of the equality constraints at the current point x_k (including linear constraints).

In order to treat NCO with a sequence of BC$_k$ subproblems, we convert the nonlinear inequality constraints to equalities to obtain

$$\boxed{\begin{array}{lll} \text{NCO}' & \underset{x,\,s}{\text{minimize}} & \phi(x) \\ & \text{subject to} & c(x) - s = 0, \quad Ax \geq b, \quad \ell \leq x \leq u, \quad s \geq 0 \end{array}}$$

with corresponding subproblems (including linear constraints)

$$\boxed{\begin{array}{lll} \text{BC}_k{}' & \underset{x,\,s}{\text{minimize}} & L(x, y_k, \rho_k) = \phi(x) - y_k^T(c(x) - s) + \tfrac{1}{2}\rho_k\|c(x) - s\|^2 \\ & \text{subject to} & Ax \geq b, \quad \ell \leq x \leq u, \quad s \geq 0. \end{array}}$$

We now introduce variables $r = -(c(x) - s)$ into BC$_k{}'$ to obtain the *nonlinearly constrained Lagrangian* (NCL) subproblem

$$\boxed{\begin{array}{lll} \text{NC}_k & \underset{x,\,r}{\text{minimize}} & \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k\|r\|^2 \\ & \text{subject to} & c(x) + r \geq 0, \quad Ax \geq b, \quad \ell \leq x \leq u, \end{array}}$$

in which r serves to make the nonlinear constraints independent. (If NCO includes an equality $c_i(x) = 0$, NC$_k$ would contain $c_i(x) + r_i = 0$.) Assuming existence of finite multipliers and feasibility, for $\rho_k > 0$ and larger than a certain finite value, the NCL subproblems should cause y_k to approach y^* and most of the solution $(x_k^*, r_k^*, y_k^*, z_k^*)$ of NC$_k$ to approach (x^*, y^*, z^*), with r_k^* approaching zero.

Problem NC$_k$ is analogous to Friedlander and Orban's formulation for convex quadratic programs [7, Eq. (3.2)]. See also Arreckx and Orban [2], where the motivation is the same as here, achieving reliability when the nonlinear constraints do not satisfy LICQ.

Note that for general problems NECB, the BCL and LCL subproblems contain linear constraints (bounds only, or linearized constraints and bounds). Our NCL formulation retains nonlinear constraints in the NC$_k$ subproblems, but simplifies them by ensuring that they satisfy LICQ. On large problems, the additional variables $r \in \mathbb{R}^m$ in NC$_k$ may be detrimental to active-set solvers like MINOS or SNOPT [11] because they increase the number of degrees of freedom (superbasic variables). Fortunately, they are easily accommodated by interior methods, as our numerical results show for IPOPT [12, 19]. We expect the same to be true for KNITRO [3, 14]. These solvers are most effective when second derivatives are available, as they are for our AMPL model.

2.1 The BCL Algorithm

The LANCELOT BCL method is summarized in Algorithm BCL. Each subproblem BC$_k$ is solved with a specified optimality tolerance ω_k, generating an iterate x_k^* and the associated Lagrangian gradient $z_k^* \equiv \nabla L(x_k^*, y_k, \rho_k)$. If $\|c(x_k^*)\|$ is sufficiently small, the iteration is regarded as "successful" and an update to y_k is computed from x_k^*. Otherwise, y_k is not altered but ρ_k is increased.

Key properties are that the subproblems are solved inexactly, the penalty parameter is increased only finitely often, and the multiplier estimates y_k need not be assumed bounded. Under certain conditions, all iterations are eventually successful, the ρ_k's remain constant, the iterates converge superlinearly, and the algorithm terminates in a finite number of iterations [4].

Algorithm 1 BCL (Bound-Constrained Lagrangian Method for NECB)

```
1:  procedure BCL($x_0, y_0, z_0$)
2:      Set penalty parameter $\rho_1 > 0$, scale factor $\tau > 1$, and constants $\alpha, \beta > 0$ with $\alpha < 1$.
3:      Set positive convergence tolerances $\eta_*, \omega_* \ll 1$ and infeasibility tolerance $\eta_1 > \eta_*$.
4:      $k \leftarrow 0$, converged $\leftarrow$ false
5:      repeat
6:          $k \leftarrow k + 1$
7:          Choose optimality tolerance $\omega_k > 0$ such that $\lim_{k\to\infty} \omega_k \le \omega_*$.
8:          Find $(x_k^*, z_k^*)$ that solves BC$_k$ to within $\omega_k$.
9:          if $\|c(x_k^*)\| \le \max(\eta_*, \eta_k)$ then
10:             $y_k^* \leftarrow y_k - \rho_k c(x_k^*)$
11:             $x_k \leftarrow x_k^*,\ y_k \leftarrow y_k^*,\ z_k \leftarrow z_k^*$        update solution estimates
12:             if $(x_k, y_k, z_k)$ solves NECB to within $\omega_*$, converged $\leftarrow$ true
13:             $\rho_{k+1} \leftarrow \rho_k$                                              keep $\rho_k$
14:             $\eta_{k+1} \leftarrow \eta_k/(1 + \rho_{k+1}^{\beta})$                     decrease $\eta_k$
15:         else
16:             $\rho_{k+1} \leftarrow \tau\rho_k$                                          increase $\rho_k$
17:             $\eta_{k+1} \leftarrow \eta_0/(1 + \rho_{k+1}^{\alpha})$                    may increase or decrease $\eta_k$
18:         end if
19:     until converged
20:     $x^* \leftarrow x_k,\ y^* \leftarrow y_k,\ z^* \leftarrow z_k$
21: end procedure
```

Note that at step 8 of Algorithm BCL, the inexact minimization would typically use the initial guess (x_k^*, z_k^*). However, other initial points are possible. At step 12, we say that (x_k, y_k, z_k) solves NECB to within ω_* if the largest dual infeasibility is smaller than ω_*.

Algorithm 2 NCL (Nonlinearly Constrained Lagrangian Method for NCO)

```
1: procedure NCL($x_0, r_0, y_0, z_0$)
2:     Set penalty parameter $\rho_1 > 0$, scale factor $\tau > 1$, and constants $\alpha, \beta > 0$ with $\alpha < 1$.
3:     Set positive convergence tolerances $\eta_*, \omega_* \ll 1$ and infeasibility tolerance $\eta_1 > \eta_*$.
4:     $k \leftarrow 0$, converged $\leftarrow$ false
5:     repeat
6:         $k \leftarrow k + 1$
7:         Choose optimality tolerance $\omega_k > 0$ such that $\lim_{k\to\infty} \omega_k \le \omega_*$.
8:         Find $(x_k^*, r_k^*, y_k^*, z_k^*)$ that solves NC$_k$ to within $\omega_k$.
9:         if $\|r_k^*\| \le \max(\eta_*, \eta_k)$ then
10:            $y_k^* \leftarrow y_k + \rho_k r_k^*$
11:            $x_k \leftarrow x_k^*,\ r_k \leftarrow r_k^*,\ y_k \leftarrow y_k^*,\ z_k \leftarrow z_k^*$        update solution estimates
12:            if $(x_k, y_k, z_k)$ solves NCO to within $\omega_*$, converged $\leftarrow$ true
13:            $\rho_{k+1} \leftarrow \rho_k$        keep $\rho_k$
14:            $\eta_{k+1} \leftarrow \eta_k/(1 + \rho_{k+1}^\beta)$        decrease $\eta_k$
15:        else
16:            $\rho_{k+1} \leftarrow \tau\rho_k$        increase $\rho_k$
17:            $\eta_{k+1} \leftarrow \eta_0/(1 + \rho_{k+1}^\alpha)$        may increase or decrease $\eta_k$
18:        end if
19:    until converged
20:    $x^* \leftarrow x_k,\ r^* \leftarrow r_k,\ y^* \leftarrow y_k,\ z^* \leftarrow z_k$
21: end procedure
```

2.2 *The NCL Algorithm*

To derive a stabilized algorithm for problem NCO, we modify Algorithm BCL by introducing r and replacing the subproblems BC$_k$ by NC$_k$. The resulting method is summarized in Algorithm NCL. The update to y_k becomes $y_k^* \leftarrow y_k - \rho_k(c(x_k^*) - s_k^*) = y_k + \rho_k r_k^*$, the value satisfied by an optimal y_k^* for subproblem NC$_k$. Step 8 of Algorithm NCL would typically use $(x_k^*, r_k^*, y_k^*, z_k^*)$ as initial guess, and that is what we use in our implementation below.

3 An Application: Optimal Tax Policy

Some challenging test cases arise from the tax policy models described in [13]. With $x = (c, y)$, they take the form

$$\boxed{\begin{array}{llll} \text{TAX} & \underset{c,\,y}{\text{maximize}} & \sum_i \lambda_i U^i(c_i, y_i) & \\ & \text{subject to} & U^i(c_i, y_i) - U^i(c_j, y_j) \ge 0 & \text{for all } i, j \\ & & \lambda^T(y - c) \ge 0 & \\ & & c,\ y \ge 0, & \end{array}}$$

where c_i and y_i are the consumption and income of taxpayer i, and λ is a vector of positive weights. The utility functions $U^i(c_i, y_i)$ are each of the form

$$U(c, y) = \frac{(c - \alpha)^{1-1/\gamma}}{1 - 1/\gamma} - \psi \frac{(y/w)^{1/\eta+1}}{1/\eta + 1},$$

where w is the wage rate and α, γ, ψ, and η are taxpayer heterogeneities. More precisely, the utility functions are of the form

$$U^{i,j,k,g,h}(c_{p,q,r,s,t}, y_{p,q,r,s,t}) = \frac{(c_{p,q,r,s,t} - \alpha_k)^{1-1/\gamma_h}}{1 - 1/\gamma_h} - \psi_g \frac{(y_{p,q,r,s,t}/w_i)^{1/\eta_j+1}}{1/\eta_j + 1},$$

where (i, j, k, g, h) and (p, q, r, s, t) run over na wage types, nb elasticities of labor supply, nc basic need types, nd levels of distaste for work, and ne elasticities of demand for consumption, with na, nb, nc, nd, ne determining the size of the problem, namely $m = T(T - 1)$ nonlinear constraints, $n = 2T$ variables, with $T := na \times nb \times nc \times nd \times ne$.

Table 1 summarizes results for a 4D example ($ne = 1$ and $\gamma_1 = 1$). The first term of $U(c, y)$ becomes $\log(c - \alpha)$, the limit as $\gamma \to 1$. Problem NCO and Algorithm NCL were formulated in the AMPL modeling language [6]. The solvers SNOPT [11] and IPOPT [19] were unable to solve NCO itself, but Algorithm NCL was successful with IPOPT solving the subproblems NC$_k$. We use a default configuration of IPOPT with MUMPS [1] as symmetric indefinite solver to compute search directions. We set the optimality tolerance for IPOPT to $\omega_k = \omega_* = 10^{-6}$ throughout and specified warm starts for $k \geq 2$ using options warm_start_init_point=yes and mu_init=1e-4. These options greatly improved the performance of IPOPT on each subproblem compared to cold starts, for which mu_init=0.1. It is helpful that only the objective function of NC$_k$ changes with k.

Table 1 NCL results on a 4D example with $na, nb, nc, nd = 11, 3, 3, 2$, giving $m = 39006$, $n = 395$. Itns refers to IPOPT's primal-dual interior point method, and time is seconds on an Apple iMac with 2.93 GHz Intel Core i7

k	ρ_k	η_k	$\Vert r_k^*\Vert_\infty$	$\phi(x_k^*)$	Itns	Time
1	10^2	10^{-2}	3.1e − 03	−2.1478532e + 01	125	42.8
2	10^2	10^{-3}	1.3e − 03	−2.1277587e + 01	18	6.5
3	10^3	10^{-3}	6.6e − 04	−2.1177152e + 01	27	9.1
4	10^3	10^{-4}	5.5e − 04	−2.1110210e + 01	31	10.8
5	10^4	10^{-4}	2.9e − 04	−2.1066664e + 01	57	24.3
6	10^5	10^{-4}	6.5e − 05	−2.1027152e + 01	75	26.8
7	10^5	10^{-5}	5.2e − 05	−2.1018896e + 01	130	60.9
8	10^6	10^{-5}	9.3e − 06	−2.1015295e + 01	159	81.8
9	10^6	10^{-6}	2.0e − 06	−2.1014808e + 01	139	70.0
10	10^7	10^{-6}	2.1e − 07	−2.1014800e + 01	177	97.6

For this example, problem NCO has $m = 39{,}006$ nonlinear inequality constraints and one linear constraint in $n = 395$ variables $x = (c, y)$, and nonnegativity bounds. Subproblem NC$_k$ has 39,007 constraints and 39,402 variables when r is included. Fortunately, r does not affect the complexity of each IPOPT iteration, but greatly improves stability. In contrast, active-set methods like MINOS and SNOPT are very inefficient on the NC$_k$ subproblems because the large number of inequality constraints leads to thousands of minor iterations, and the presence of r (with no bounds) leads to thousands of superbasic variables. About $3.2n$ constraints were within 10^{-6} of being active.

Table 2 summarizes results for a 5D example. The NC$_k$ subproblems have $m = 32{,}220$ nonlinear constraints and $n = 360$ variables, leading to 32,581 variables including r. Again the options warm_start_init_point=yes and mu_init=1e-4 for $k \geq 2$ led to good performance by IPOPT on each subproblem. About $3n$ constraints were within 10^{-6} of being active.

Table 2 NCL results on a 5D example with $na, nb, nc, nd, ne = 5, 3, 3, 2, 2$, giving $m = 32220$, $n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	Itns	Time
1	10^2	10^{-2}	7.0e − 03	−4.2038075e + 02	95	41.1
2	10^2	10^{-3}	4.1e − 03	−4.2002898e + 02	17	7.2
3	10^3	10^{-3}	1.3e − 03	−4.1986069e + 02	20	8.1
4	10^4	10^{-3}	4.4e − 04	−4.1972958e + 02	48	25.0
5	10^4	10^{-4}	2.2e − 04	−4.1968646e + 02	43	20.5
6	10^5	10^{-4}	9.8e − 05	−4.1967560e + 02	64	32.9
7	10^5	10^{-5}	6.6e − 05	−4.1967177e + 02	57	26.8
8	10^6	10^{-5}	4.2e − 06	−4.1967150e + 02	87	46.2
9	10^6	10^{-6}	9.4e − 07	−4.1967138e + 02	96	53.6

Table 3 NCL results on a 5D example with $na, nb, nc, ne, ne = 21, 3, 3, 2, 2$, giving $m = 570780$, $n = 1512$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	5.1e − 03	−1.7656816e + 03	10^{-1}	825	7763.3
2	10^2	10^{-3}	2.4e − 03	−1.7648480e + 03	10^{-4}	66	472.8
3	10^3	10^{-3}	1.3e − 03	−1.7644006e + 03	10^{-4}	106	771.3
4	10^4	10^{-3}	3.8e − 04	−1.7639491e + 03	10^{-5}	132	1347.0
5	10^4	10^{-4}	3.2e − 04	−1.7637742e + 03	10^{-5}	229	2450.9
6	10^5	10^{-4}	8.6e − 05	−1.7636804e + 03	10^{-6}	104	1096.9
7	10^5	10^{-5}	4.9e − 05	−1.7636469e + 03	10^{-6}	143	1633.4
8	10^6	10^{-5}	1.5e − 05	−1.7636252e + 03	10^{-7}	71	786.1
9	10^7	10^{-5}	2.8e − 06	−1.7636196e + 03	10^{-7}	67	725.7
10	10^7	10^{-6}	5.1e − 07	−1.7636187e + 03	10^{-8}	18	171.0

For much larger problems of this type, we found that it was helpful to reduce mu_init more often, as illustrated in Table 3. The NC$_k$ subproblems here have $m = 570,780$ nonlinear constraints and $n = 1512$ variables, leading to $572{,}292$ variables including r. Note that the number of NCL iterations is stable ($k \le 10$), and IPOPT performs well on each subproblem with decreasing mu_init. This time about $6.6n$ constraints were within 10^{-6} of being active.

Note that the LANCELOT approach allows early subproblems to be solved less accurately [4]. It may save time to set $\omega_k = \eta_k$ (say) rather than $\omega_k = \omega_*$ throughout.

4 Conclusions

This work has been illuminating in several ways as we sought to improve our ability to solve examples of problem TAX.

- Small examples of the tax model solve efficiently with MINOS and SNOPT, but eventually fail to converge as the problem size increases.
- IPOPT also solves small examples efficiently, but eventually starts requesting additional memory for the MUMPS sparse linear solver. The solver may freeze, or the iterations may diverge.
- The NC$_k$ subproblems are not suitable for MINOS or SNOPT because of the large number of variables (x, r) and the resulting number of superbasic variables (although warm starts are natural).
- It is often said that interior methods cannot be warm started. Nevertheless, IPOPT has several runtime options that have proved to be extremely helpful for implementing Algorithm NCL. For the results obtained here, it has been sufficient to say that warm starts are wanted for $k > 1$, and that the IPOPT barrier parameter should be initialized at decreasing values for later k (where only the objective of subproblem NC$_k$ changes with k).
- The numerical examples of Sect. 3 had $3n$, $3n$, and $6.6n$ constraints essentially active at the solution, yet were solved successfully. They suggest that the NCL approach with an interior method as subproblem solver can overcome LICQ difficulties on problems that could not be solved directly.

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Appendix A AMPL Models, Data, and Scripts

Algorithm NCL has been implemented in the AMPL modeling language [6] and tested on problem TAX. The following sections list each relevant file. The files are available from [17].

A.1 Tax Model

File `pTax5Dncl.mod` codes subproblem NC_k for problem TAX with five parameters $w, \eta, \alpha, \psi, \gamma$, using $\mu := 1/\eta$. Note that for $U(c, y)$ in the objective and constraint functions, the first term $(c-\alpha)^{1-1/\gamma}/(1-1/\gamma)$ is replaced by a piecewise-smooth function that is defined for all values of c and α (see [13]).

Primal regularization $\frac{1}{2}\delta\|(c, y)\|^2$ with $\delta = 10^{-8}$ is added to the objective function to promote uniqueness of the minimizer. The vector r is called `R` to avoid a clash with subscript `r`.

```
# pTax5Dncl.mod

# Define parameters for agents (taxpayers)
param na;                    # number of types in wage
param nb;                    # number of types in eta
param nc;                    # number of types in alpha
param nd;                    # number of types in psi
param ne;                    # number of types in gamma
set A := 1..na;              # set of wages
set B := 1..nb;              # set of eta
set C := 1..nc;              # set of alpha
set D := 1..nd;              # set of psi
set E := 1..ne;              # set of gamma
set T = {A,B,C,D,E};         # set of agents

# Define wages for agents (taxpayers)
param wmin;                  # minimum wage level
param wmax;                  # maximum wage level
param w {A};                 # i, wage vector
param mu{B};                 # j, mu = 1/eta# mu vector
param mu1{B};                # mu1[j] = mu[j] + 1
param alpha{C};              # k, ak vector for utility
param psi{D};                # g
param gamma{E};              # h
param lambda{A,B,C,D,E};     # distribution density
param epsilon;
param primreg      default 1e-8;   # Small primal
                                     regularization

var y{(i,j,k,g,h) in T} >= 0.1;    # consumption for
                                     tax payer
                                     (i,j,k,g,h)
```

```
var y{(i,j,k,g,h) in T} >= 0.1;  # income  for
                                   tax payer
                                   (i,j,k,g,h)
var R{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
      !(i=p and j=q and k=r and g=s and h=t)}
      >= -1e+20, <= 1e+20;

param kmax       default 20;     # limit on
                                   NCL itns
param rhok       default 1e+2;   # augmented
                                   Lagrangian penalty
                                   parameter
param rhofac     default 10.0;   # increase
                                   factor
param rhomax     default 1e+8;   # biggest rhok
param etak       default 1e-2;   # opttol for
                                   augmented
                                   Lagrangian loop
param etafac     default  0.1;   # reduction factor for
                                    opttol
param etamin     default 1e-8;   # smallest etak
param rmax       default    0;   # max r (for printing)

param rmin       default    0;   # min r (for printing)

param rnorm      default    0;   # ||r||_inf
param rtol       default 1e-6;   # quit if biggest
                                    |r_i| <= rtol

param nT         default    1;   # nT = na*nb
                                   *nc*nd*ne
param m          default    1;   # nT*(nT-1)
                                   = no. of nonlinear
                                    constraints
param n          default    1;   # 2*nT
                                   = no. of nonlinear
                                   variables

param ck{(i,j,k,g,h) in T} default 0;
            # current variable c
param yk{(i,j,k,g,h) in T} default 0;
            # current variable y
param rk{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
            # current variable r = - (c(x) - s)
   !(i=p and j=q and k=r and g=s and h=t)} default 0;
param dk{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
            # current dual variables (y_k)
   !(i=p and j=q and k=r and g=s and h=t)} default 0;

minimize f:
   sum{(i,j,k,g,h) in T}
   (
      (if c[i,j,k,g,h] - alpha[k] >= epsilon then
          - lambda[i,j,k,g,h] *
               ((c[i,j,k,g,h] - alpha[k])
                ^(1-1/gamma[h]) / (1-1/gamma[h])
                - psi[g]*(y[i,j,k,g,h]/w[i])
                ^mu1[j] / mu1[j])
       else
          - lambda[i,j,k,g,h] *
         (-   0.5/gamma[h] * epsilon^(-1/gamma[h]-1)
          * (c[i,j,k,g,h] - alpha[k])^2
          + ( 1+1/gamma[h])* epsilon^(-1/gamma[h]  )
          * (c[i,j,k,g,h] - alpha[k])
          + (1/(1-1/gamma[h]) - 1 - 0.5/gamma[h])
          * epsilon^(1-1/gamma[h])
                - psi[g]*(y[i,j,k,g,h]/w[i])
                ^mu1[j] / mu1[j])
      )
   + 0.5 * primreg * (c[i,j,k,g,h]^2
   + y[i,j,k,g,h]^2)
   )
 + sum{(i,j,k,g,h) in T, (p,q,r,s,t) in
 T: !(i=p and j=q and k=r and g=s and h=t)}
       (dk[i,j,k,g,h,p,q,r,s,t]
        * R[i,j,k,g,h,p,q,r,s,t]
                    + 0.5 * rhok
                    * R[i,j,k,g,h,p,q,r,s,t]^2);

subject to

Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
            !(i=p and j=q and k=r and g=s and h=t)}:
   (if c[i,j,k,g,h] - alpha[k] >= epsilon then
      (c[i,j,k,g,h] - alpha[k])
      ^(1-1/gamma[h]) / (1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])
       ^mu1[j] / mu1[j]
    else
       -  0.5/gamma[h] *epsilon^(-1/gamma[h]-1)
       *(c[i,j,k,g,h] - alpha[k])^2
       + (1+1/gamma[h])*epsilon^(-1/gamma[h]  )
       *(c[i,j,k,g,h] - alpha[k])
       + (1/(1-1/gamma[h]) - 1 - 0.5/gamma[h])
       *epsilon^(1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j]
       / mu1[j]
   )
 - (if c[p,q,r,s,t] - alpha[k] >= epsilon then
      (c[p,q,r,s,t] - alpha[k])^(1-1/gamma[h])
      / (1-1/gamma[h])
       - psi[g]*(y[p,q,r,s,t]/w[i])^mu1[j] / mu1[j]
    else
        -  0.5/gamma[h] *epsilon^(-1/gamma[h]-1)
        *(c[p,q,r,s,t] - alpha[k])^2
        + (1+1/gamma[h])*epsilon^(-1/gamma[h]  )
        *(c[p,q,r,s,t] - alpha[k])
        + (1/(1-1/gamma[h]) - 1 - 0.5/gamma[h])
        *epsilon^(1-1/gamma[h])
        - psi[g]*(y[p,q,r,s,t]/w[i])^mu1[j] / mu1[j]
   )
 + R[i,j,k,g,h,p,q,r,s,t] >= 0;

Technology:
   sum{(i,j,k,g,h) in T} lambda[i,j,k,g,h]
   *(y[i,j,k,g,h] - c[i,j,k,g,h]) >= 0;
```

A.2 Tax Model Data

File `pTax5Dncl.dat` provides data for a specific problem.

```
# pTax5Dncl.dat

data;

let na := 5;
let nb := 3;
let nc := 3;
let nd := 2;
let ne := 2;

# Set up wage dimension intervals
let wmin := 2;
let wmax := 4;
let {i in A}  w[i]   := wmin + ((wmax-wmin)
/(na-1))*(i-1);

data;

param  mu :=
    1   0.5
    2   1
    3   2 ;

# Define mu1
let {j in B} mu1[j] := mu[j] + 1;

data;

param alpha :=
    1   0
    2   1
    3   1.5;

param psi :=
    1   1
    2   1.5;

param gamma :=
    1   2
    2   3;

# Set up 5 dimensional distribution
let {(i,j,k,g,h) in T} lambda[i,j,k,g,h] := 1;

# Choose a reasonable epsilon
let epsilon := 0.1;
```

A.3 Initial Values

File `pTax5Dinitial.run` solves a simplified model to compute starting values for Algorithm NCL. The nonlinear inequality constraints are removed, and $y = c$ is enforced. This model solves easily with MINOS or SNOPT on all cases tried. Solution values are output to file `p5Dinitial.dat`.

```
# pTax5Dinitial.run

# Define parameters for agents (taxpayers)
param na := 5;          # number of types in wage
param nb := 3;          # number of types in eta
param nc := 3;          # number of types in alpha
param nd := 2;          # number of types in psi
param ne := 2;          # number of types in gamma
set A := 1..na;         # set of wages
set B := 1..nb;         # set of eta
set C := 1..nc;         # set of alpha
set D := 1..nd;         # set of psi
set E := 1..ne;         # set of gamma
set T = {A,B,C,D,E};    # set of agents

# Define wages for agents (taxpayers)
param  wmin := 2;          # minimum wage level
param  wmax := 4;          # maximum wage level
param  w {i in A} := wmin + ((wmax-wmin)
/(na-1))*(i-1);  # wage vector

# Choose a reasonable epsilon
```

```
param epsilon := 0.1;

# mu vector
param mu {B};              # mu = 1/eta
param mu1{B};              # mu1[j] = mu[j] + 1
param alpha {C};
param gamma {E};
param psi {D};

var c {(i,j,k,g,h) in T} >= 0.1;
var y {(i,j,k,g,h) in T} >= 0.1;

maximize f: sum{(i,j,k,g,h) in T}
   if c[i,j,k,g,h] - alpha[k] >= epsilon then
     (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h])
      / (1-1/gamma[h])
      -  psi[g] * (y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
   else
      -  0.5/gamma[h] *epsilon^(-1/gamma[h]-1)
      *(c[i,j,k,g,h] - alpha[k])^2
      + (1+1/gamma[h])*epsilon^(-1/gamma[h])
      *(c[i,j,k,g,h] - alpha[k])
      + (1/(1-1/gamma[h]) -1 - 0.5/gamma[h])
      *epsilon^(1-1/gamma[h])
      -  psi[g] * (y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j];

subject to
   Budget {(i,j,k,g,h) in T}: y[i,j,k,g,h]
   - c[i,j,k,g,h] = 0;

let {(i,j,k,g,h) in T} y[i,j,k,g,h] := i+1;
let {(i,j,k,g,h) in T} c[i,j,k,g,h] := i+1;

data;

param  mu :=
    1   0.5
    2   1
    3   2 ;

# Define mu1
let {j in B} mu1[j] := mu[j] + 1;

data;

param alpha :=
    1   0
    2   1
    3   1.5;

param psi :=
    1   1
    2   1.5;

param gamma :=
    1   2
    2   3;

option solver snopt;
option show_stats 1;

option snopt_options  ' \
   summary_file=6       \
   print_file=9         \
   scale=no             \
   print_level=0        \
   major_iterations=2000\
   iterations=50000     \
   optimality_tol=1e-7  \
  *penalty=100.0        \
   superbasics_limit=3000\
   solution=yes         \
  *verify_level=3       \
';


display na,nb,nc,nd,ne;
solve;
display na,nb,nc,nd,ne;
display y,c >p5Dinitial.dat;
close p5Dinitial.dat;
```

A.4 NCL Implementation

File `pTax5Dnclipopt.run` uses files

```
pTax5Dinitial.run
pTax5Dncl.mod
pTax5Dncl.dat
pTax5Dinitial.dat
```

to implement Algorithm NCL. Subproblems NC$_k$ are solved in a loop until $\|r_k^*\|_\infty \le$ `rtol = 1e-6`, or η_k has been reduced to parameter `etamin = 1e-8`, or ρ_k has been increased to parameter `rhomax = 1e+8`. The loop variable k is called `K` to avoid a clash with subscript `k` in the model file. The definitions of `etak` and `rhok` inside the loop are simpler than (but similar to) the settings of η_k and ρ_k in Algorithm 2.

Optimality tolerance $\omega_k = \omega_* = 10^{-6}$ is used throughout to ensure that the solution of the final subproblem NC_k will be close to a solution of the original problem if $\|r_k^*\|_\infty$ is small enough for the final k ($\|r_k^*\|_\infty \le$ `rtol = 1e-6`).

IPOPT is used to solve each subproblem NC_k, with runtime options set to implement increasingly warm starts.

```
# pTax5Dnclipopt.run

reset;   model pTax5Dinitial.run;
reset;   model pTax5Dncl.mod;
data  pTax5Dncl.dat;
data; var include p5Dinitial.dat;

model;
option solver ipopt;
option show_stats 1;

option ipopt_options '\
 dual_inf_tol=1e-6     \
 max_iter=5000         \
';

option opt2 $ipopt_options ' warm_start_init_point
=yes';

# NCL method.
# kmax, rhok, rhofac, rhomax, etak, etafac,
  etamin, rtol
# are defined in the .mod file.

printf "NCLipopt log for pTax5D\n" > 5DNCLipopt.log;
display na, nb, nc, nd, ne, primreg  > 5DNCLipopt.log;
printf "   k      rhok      etak     rnorm
Obj\n" > 5DNCLipopt.log;

for {K in 1..kmax}
{  display na, nb, nc, nd, ne, primreg, K, kmax,
   rhok, etak;
   if K == 2 then {option ipopt_options
   $opt2 ' mu_init=1e-4'};
   if K == 4 then {option ipopt_options
   $opt2 ' mu_init=1e-5'};
   if K == 6 then {option ipopt_options
   $opt2 ' mu_init=1e-6'};
   if K == 8 then {option ipopt_options
   $opt2 ' mu_init=1e-7'};
   if K ==10 then {option ipopt_options
   $opt2 ' mu_init=1e-8'};

   display $ipopt_options;
   solve;

   let rmax := max({(i,j,k,g,h) in T, (p,q,r,s,t)
   in T:
      !(i=p and j=q and k=r and g=s and h=t)}
      R[i,j,k,g,h,p,q,r,s,t]);
   let rmin := min({(i,j,k,g,h) in T, (p,q,r,s,t)
   in T:
      !(i=p and j=q and k=r and g=s and h=t)}
      R[i,j,k,g,h,p,q,r,s,t]);
   display na, nb, nc, nd, ne, primreg, K, rhok,
   etak, kmax;
   display K, kmax, rmax, rmin;
   let rnorm := max(abs(rmax), abs(rmin));
   # ||r||_inf

   printf "%4i %9.1e %9.1e %9.1e %15.7e\n",
   K, rhok, etak, rnorm, f >> 5DNCLipopt.log;
   close 5DNCLipopt.log;

   if rnorm <= rtol then
   { printf "Stopping: rnorm is small\n";
     display K, rnorm; break; }

   if rnorm <= etak then # update dual estimate dk;
   save new solution
   {let {(i,j,k,g,h) in T, (p,q,r,s,t) in T:
          !(i=p and j=q and k=r and g=s and h=t)}
             dk[i,j,k,g,h,p,q,r,s,t] :=
             dk[i,j,k,g,h,p,q,r,s,t] + rhok
             *R[i,j,k,g,h,p,q,r,s,t];
    let {(i,j,k,g,h) in T} ck[i,j,k,g,h] :=
    c[i,j,k,g,h];
    let {(i,j,k,g,h) in T} yk[i,j,k,g,h] :=
    y[i,j,k,g,h];
    display K, etak;
    if  etak == etamin then { printf "Stopping:
    etak = etamin\n"; break; }
    let etak := max(etak*etafac, etamin);
    display etak;
   }
   else # keep previous solution; increase rhok
   { let {(i,j,k,g,h) in T} c[i,j,k,g,h] :=
     ck[i,j,k,g,h];
     let {(i,j,k,g,h) in T} y[i,j,k,g,h] :=
     yk[i,j,k,g,h];
     display K, rhok;
     if  rhok == rhomax then { printf "Stopping:
     rhok = rhomax\n"; break; }
     let rhok := min(rhok*rhofac, rhomax);
```

```
      display rhok;
    }
}

display c,y;  display na, nb, nc, nd, ne, primreg,
rhok, etak, rnorm;

# Count how many constraint are close to being active.
data;
let nT   := na*nb*nc*nd*ne;    let m := nT*(nT-1);
let n := 2*nT;
let etak := 1.0001e-10;
printf "\n m = %8i\n n = %8i\n", m, n >>
5DNCLipopt.log;
printf "\n Constraints within tol of being
active\n\n" >> 5DNCLipopt.log;
printf "    tol      count    count/n\n" >>
5DNCLipopt.log;

for {K in 1..10}
{ let kmax := card{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
                   !(i=p and j=q and k=r and g=s
                   and h=t)
                   and Incentive[i,j,k,g,h,p,q,r,s,t].
                   slack <= etak};
  printf "%9.1e %8i %8.1f\n", etak, kmax,
  kmax/n >> 5DNCLipopt.log;
  let etak := etak*10.0;
}
printf "Created 5DNCLipopt.log\n";
```

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