High Frequency Trading

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June 1st 2021

Introduction

Long Term Goals: Design and implement a trading strategy based on high frequency stocks data.

Data:

```
[112]: df.head(10)
```

[112]:

	datetime	bid	bs	ask	as	mid	imb	wmid	spread	time
0	2021-01-05 09:00:00.000	128.40	988	128.42	1561	128.410	0.387603	128.407752	0.02	0
1	2021-01-05 09:00:00.100	128.40	1088	128.42	1561	128.410	0.410721	128.408214	0.02	1
2	2021-01-05 09:00:00.200	128.40	1088	128.42	1561	128.410	0.410721	128.408214	0.02	2
3	2021-01-05 09:00:00.300	128.40	1088	128.42	1561	128.410	0.410721	128.408214	0.02	3
4	2021-01-05 09:00:00.400	128.40	1088	128.42	1561	128.410	0.410721	128.408214	0.02	4
5	2021-01-05 09:00:00.500	128.39	12	128.42	1661	128.405	0.007173	128.390215	0.03	5
6	2021-01-05 09:00:00.600	128.39	12	128.42	1661	128.405	0.007173	128.390215	0.03	6
7	2021-01-05 09:00:00.700	128.39	12	128.42	1661	128.405	0.007173	128.390215	0.03	7
8	2021-01-05 09:00:00.800	128.39	12	128.42	1661	128.405	0.007173	128.390215	0.03	8
9	2021-01-05 09:00:00.900	128.39	12	128.42	1661	128.405	0.007173	128.390215	0.03	9

Figure: A glimpse into AAPL (2021-01-05)

Introduction (2)

Question: From the point of view of a market maker, what is the fair price given the state of the order book?

If we know that fair price $\hat{P}$, we are able to place an order (P^b_{t+1}, P^a_{t+1}) such that:

$$P^b_t \leq P^b_{t+1} \leq \hat{P} \leq P^a_{t+1} \leq P^a_t \quad (1)$$

Thus, we provide liquidity all the while being covered against price variations.

State of the Art

- Mid-Price

$$M_t = \frac{P_t^b + P_t^a}{2} \tag{2}$$

- Weighted Mid-Price

$$W_t = I_t P_t^a + (1 - I_t) P_t^b \tag{3}$$

$$I_t = \frac{Q_t^b}{Q_t^a + Q_t^b} \tag{4}$$

where P_t^a (resp. P_t^b) denotes the price at best ask (resp. best bid) and Q_t^a (resp. Q_t^b) denotes the volume at best ask (resp. at best bid)

State of the Art (2)

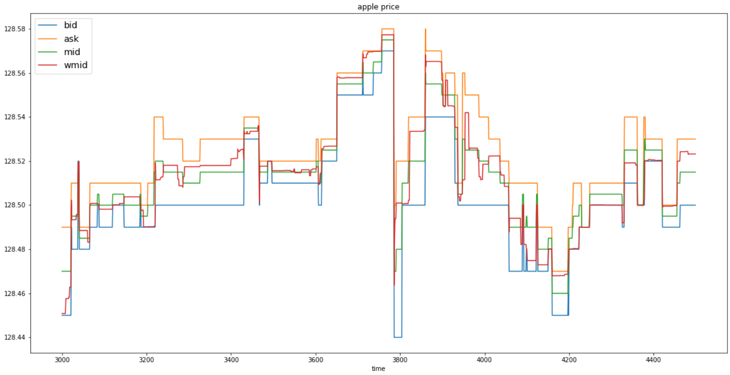


Figure: Bid, ask, mid and weighted mid prices AAPL stock (2021-01-05)

The Micro Price

Goal

Build a fair estimator of the price P_t of the stock at time t.

Definitions

- Times when the mid-price changes:

$$\tau_1 = \inf\{u > t, M_u - M_{u^-} \neq 0\} \quad (5)$$

$$\tau_{i+1} = \inf\{u > \tau_i, M_u - M_{u^-} \neq 0\} \quad (6)$$

- Micro-price:

$$P_t^{micro} := \lim_{i \to \infty} \mathrm{E}\left[M_{\tau_i} | \mathcal{F}_t\right] \quad (7)$$

Interpretation: If we are at time t, we consider that the fair price is the conditional expectation of future mid-prices based on the current state of the order book (analogy with Black-Scholes theory).

The Micro Price (2)

Assumptions

- The information in the order book is determined by the processes of the mid, the imbalance and the spread:

$$\mathcal{F}_t = \sigma(M_s, I_s, S_s; s \leq t) \tag{8}$$

- Mid price increments are independent from mid-price level:

$$\begin{aligned} \mathrm{E}[M_{\tau_{i+1}} - M_{\tau_i} | M_t = M, I_t = I, S_t = S] = \\ \mathrm{E}\left[M_{\tau_{i+1}} - M_{\tau_i} | I_t = I, S_t = S\right] \end{aligned}$$

The Micro Price (3)

Theorem:

Given these two assumptions, the prediction of the i^{th} mid-price can be written as:

$$\mathrm{E}\left[M_{\tau_i} | \mathcal{F}_t\right] = M_t + \sum_{k=1}^{i} g^k(I_t, S_t) \tag{9}$$

where

$$g^1(I, S) = \mathrm{E}\left[M_{\tau_1} - M_t | I_t = I, S_t = S\right] \tag{10}$$

and

$$g^{i+1}(I, S) = \mathrm{E}\left[g^i(I_{\tau_1}, S_{\tau_1}) | I_t = I, S_t = S\right] \tag{11}$$

The Micro Price (4)

The finite-space model

We denote by $X_t := (I_t, S_t)$ the state of the order book. For a finite number of possible values of the imbalance I_t and spread S_t, we can express the micro-price as:

$$P_t^{micro} = M_t + \sum_{k=1}^{\infty} B^k G^1 \tag{12}$$

where:

$$B := (I - Q)^{-1} T \tag{13}$$

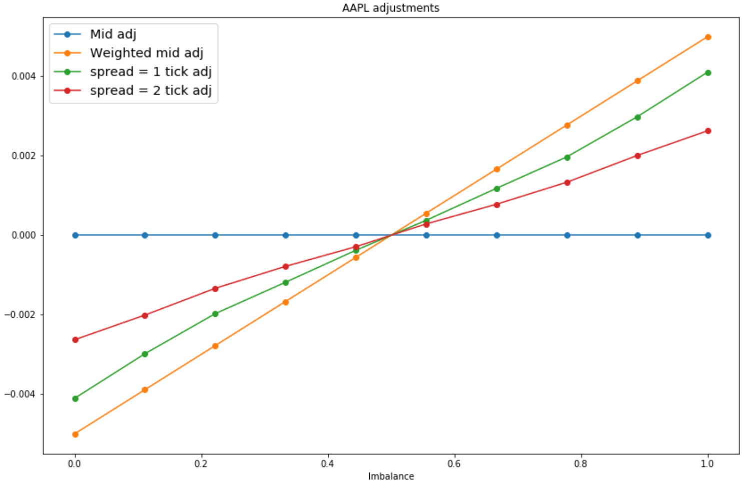
$$G^1 := (I - Q)^{-1} RK \tag{14}$$

$$Q_{xy} := \mathrm{P(M_{t+1} - M_t = 0 \cap X_{t+1} = y | X_t = x)} \tag{15}$$

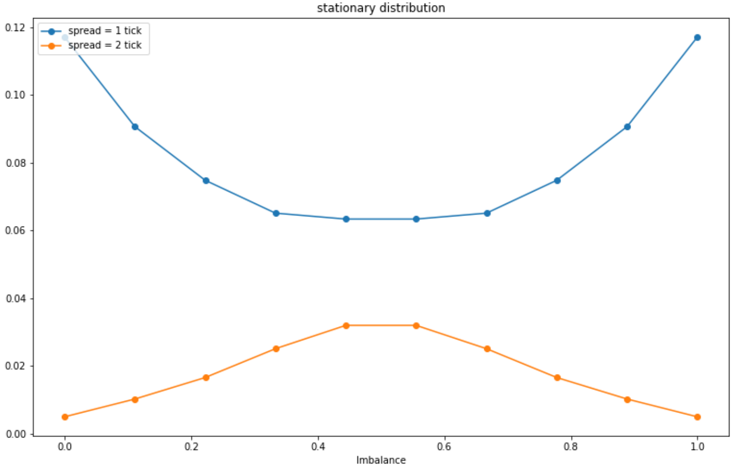
$$T_{xy} := \mathrm{P(M_{t+1} - M_t \neq 0 \cap X_{t+1} = y | X_t = x)} \tag{16}$$

$$R_{xk} := \mathrm{P(M_{t+1} - M_t = k | X_t = x)} \tag{17}$$

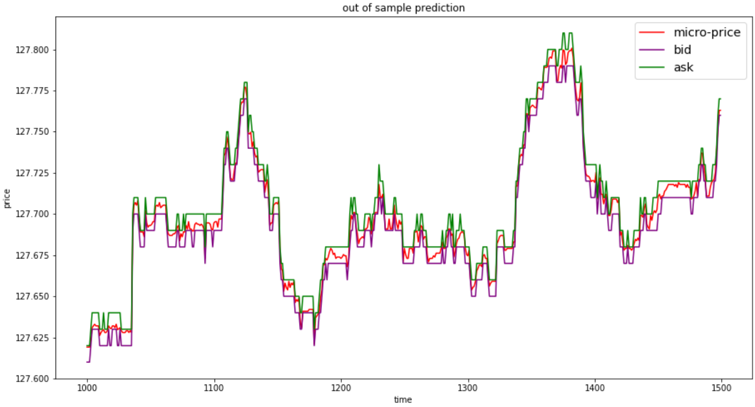
Results adjustment



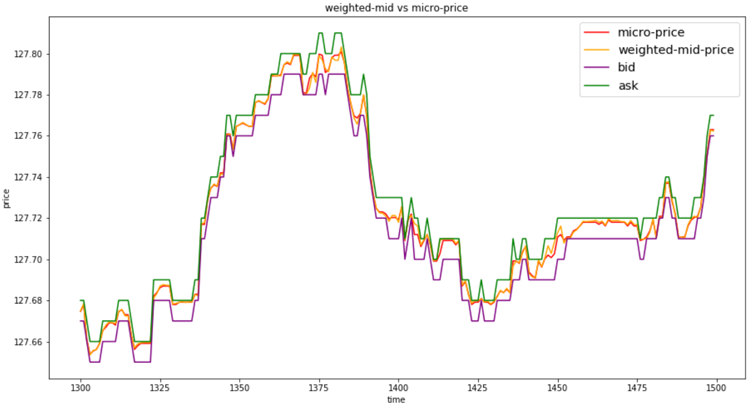
Results state distribution



Out of sample Predictions



Out of sample wMid vs MicroP



Second Strategy: With inventory

Assumptions

- The mid price is a geometric brownian motion with volatility σ, the agent has no opinion on the drift or any autocorrellation structure of the stock

$$dS_u = \sigma dW_u \tag{18}$$

- The agent's value function is:

$$v(x, s, q, t) = \mathrm{E}\left[-\exp(-\gamma(x + qS_T))\right] \tag{19}$$

It can be written as $-\exp(-\gamma x)\exp(-\gamma qs)\exp\left(\frac{\gamma^2 q^2 \sigma^2 (T-t)}{2}\right)$ where x is the initial wealth in dollars.

Reservation bid and ask

Definition:

- The reservation bid r^b is defined by:

$$v(x - r^b(s,q,t), s, q+1, t) = v(x, s, q, t) \qquad (20)$$

It is the price that would make the agent indifferent between his current portfolio and his current portfolio plus one stock.

- The reservation ask r^a is defined by:

$$v(x + r^a(s,q,t), s, q-1, t) = v(x, s, q, t) \qquad (21)$$

It is the price that would make the agent indifferent between his current portfolio and his current portfolio minus one stock.

Reservation bid and ask (2)

Goal

Our goal in general will be to estimate r^a and r^b.

Analytic Solution

In our very simple framework we have:

$$r^b(s, q, t) = s - (1 + 2q)\frac{\gamma\sigma^2(T - t)}{2} \tag{22}$$

$$r^a(s, q, t) = s + (1 - 2q)\frac{\gamma\sigma^2(T - t)}{2} \tag{23}$$

In this framework, we define the **reservation price** $r(s, q, t) = s - q\gamma\sigma^2(T - t)$ as the mid of these two prices. It is an adjustment of the midprice which accounts for the inventory held by the agent.

With Limit Orders

- We allow the agent to trade the stock through limit orders that he sets around the mid-price.
- p^b denotes the bid quote ($\delta^b = s - p^b$)
- p^a denotes the ask quote ($\delta^a = s - p^a$)

Market impact

- Let us assume the agent places an order to buy Q stocks.
- Let p^Q be the price of the highest limit order executed in the trade.
- We define $\Delta p = p^Q - s$ to be the **temporary market impact** of the trade.

Trading intensity

Poisson model

- Orders to sell stock will hit the agent's buy limit order at Poisson rate $\lambda^b(\delta^b)$, a decreasing function of δ^b.
- Orders to buy stock will hit the agent's sell limit order at Poisson rate $\lambda^a(\delta^a)$, a decreasing function of δ^a.

Stochastic wealth process

The wealth and inventory are now stochastic

$$dX_t = p^a dN_t^a - p^b dN_t^b \tag{24}$$

where N_t^b is the amount of stocks bought by the agent, N_t^a is the amount of stocks sold by the agent.
The inventory is thus defined as:

$$q_t = N_t^b - N_t^a \tag{25}$$

New Optimization Problem

The objective of the agent who can set limit orders is:

$$\max_{\delta^a,\delta^b} \mathrm{E}_t \left[-\exp(-\gamma(X_T + q_T S_T)) \right] \tag{26}$$

Trading intensity

We will assume from now on that we have symmetric, exponential arrival rates:

$$\lambda^a(\delta) = \lambda^b(\delta) = Ae^{-k\delta} \tag{27}$$

which corresponds to $\Delta p \simeq ln(Q)$

Intuition on reservation price

- Basic strategy of creating symmetrical bid ask order around bid price will not work when price is trending.
- inventory risk will build up

Figure: downtrend stock

Intuition on reservation price (2)

- $r(s, q, t) = s - q\gamma\sigma^2(T - t)$
- q is our inventory, if we have positive inventory, our reservation price is lower than the mid price, vice versa for negative inventory
- gamma is our risk aversion to inventory
- T-t is the time till the end of trading day, as trading day getting closer to the end, reservation price is more "aggressive" on rebalancing inventory

Intuition on reservation price (3)

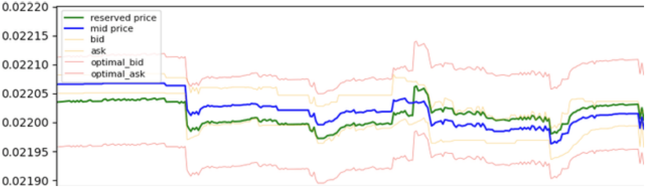


Figure: reserved price

Implementation

- 1000 simulations of different stock paths
- recall that the rate of market order is assume to be Poisson with intensity $\lambda(\delta) = A \cdot \exp(-k\delta)$
- we will place limit order at optimal ask and optimal bid and it will get executed with probability λdt
- pnl will be calculated. at each time step. profit for each limit sell order executed is $s + \delta^a$ and loss for each buy order executed is $s - \delta^b$
- **Control Strategy**: will place orders symmetrically around the mid price. it will get executed with a fixed probability p

Results

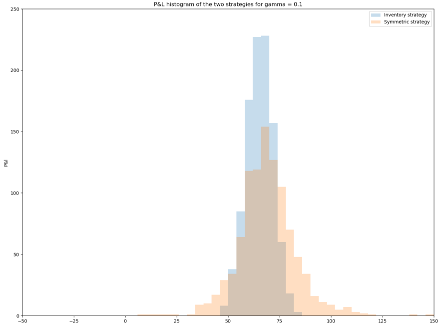


Figure: pnl with small Gamma

Strategy	Profit	Std(Profit)	Final q	Std(q)
Inventory	65.288	6.45	0.026	-0.274
Symmetric(Control)	68.932	13.88	3.017	8.40

Results (2)

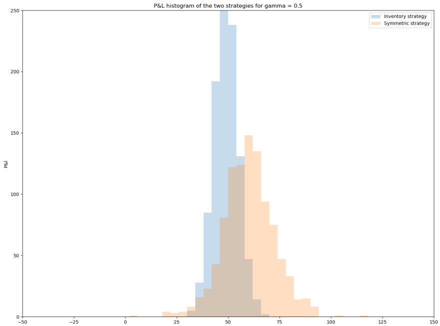


Figure: pnl with big Gamma

Strategy	Profit	Std(Profit)	Final q	Std(q)
Inventory	48.95	5.92	0.055	2.05
Symmetric(Control)	59.97	12.27	-0.123	6.982

Next Steps

- Order quantity not defined in the paper
- assumption that volatility is constant might be too strong
- order book liquidity parameter is constant