MS&E 337 Spectral Graph Theory and Algorithmic Applications

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HW#1

1. Suppose G' is a subgraph of G and λ_1 and λ_1' are the greatest eigenvalues of G and G', respectively. Prove that $\lambda_1 \geq \lambda_1'$.

2. Prove that a connected graph G with maximum eigenvalue λ_1 is bipartite if and only if $-\lambda_1$ is also an eigenvalue.

3. In this exercise, we will prove Polya's Theorem using the the connections between random walks on a graph and electric currents in a corresponding network of resistors.

   Polya's Theorem. Simple random walk on a d-dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$.

   Let G denote the lattice graph. Let $G(r)$ be the subgraph of G induced by the nodes with distance at most r from the origin. Denote by $\partial G(r)$ the set of nodes that are exactly r units away from the origin. Let $p(r)$ be the probability that a random walk on $G(r)$ starting at the origin reaches $\partial G(r)$ before returning to origin and let $p = \lim_{r\to\infty} p(r)$ be the probability for the infinite graph. An infinite walk is recurrent if and only if the limit is 0.

   To determine p electrically, we simply ground all the nodes of $\partial G(r)$, maintain the origin at one volt, and measure the current flowing into the circuit. We will have

   $$p(r) = \frac{i(r)}{2d} = \frac{1}{2dR_{\text{eff}}(r)}$$

   where d is the dimension and $R_{\text{eff}}(r)$ is the effective resistance from the origin to $\partial G(r)$.

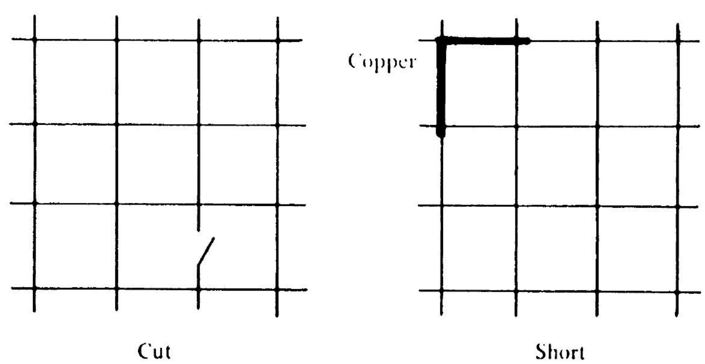
   Let $R_{\text{eff}} = \lim_{r\to\infty} R_{\text{eff}}(r)$. Then $p = 0$ iff R_{eff} is infinite.

   By using the above connection, prove Polya's theorem for one dimension.

   (a) Show that simple random walk on d-dimensional lattice is recurrent for $d = 1$.

   For $d = 2$, we need to use a shorting technique: shorting a group of nodes between the origin and outside boundary will only reduce resistance.

   (b) Show that simple random walk on d-dimensional lattice is recurrent for $d = 2$.



   To get an upper bound for higher dimensions, we need to use a cutting technique to cut the grid into trees. Compute the effective resistance of an infinite binary tree and then embed it into a 3-dimensional grid to prove the following:

(c) Show that simple random walk on d-dimensional lattice is transient for $d = 3$.

4. Then the spectral radius ρ of a matrix A is defined as:

$$\rho(A) = \max\left\{|\lambda_1|, \cdots, |\lambda_n|\right\}.$$

Prove that for every graph G,

$$\sqrt{\Delta} \leq \rho(A(G)) \leq \Delta.$$

Where Δ and $A(G)$ denote the maximum degree and the adjacency matrix of the graph respectively. Characterize the cases where equality holds.