Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and $b \in \mathbb{R}^n$ ($b \neq 0$). Let the $k$th Krylov subspace be $K_k \equiv \text{span}\{b, Ab, A^2b, \ldots, A^k b\}$, and let $(T_k, H_k, V_k)$ denote the matrices obtained by applying the Lanczos process to $(A, b)$ for $k$ steps, as defined in the notes. Assume that we use exact arithmetic and that the process terminates with $\beta_{k+1} = 0$ for some $k = \ell \leq n$.

**Exercise 1**

If $A$ has $m < n$ distinct eigenvalues, show that the Lanczos process for $(A, b)$ terminates with $\ell \leq m$. (If you wish, you may prove that the dimension of the Krylov subspaces $K_k(A, b)$ cannot exceed $m$.)

**Exercise 2**

Show that if $A$ is changed to $A - \sigma I$ for some scalar shift $\sigma$, $T_k$ becomes $T_k - \sigma I$ and $V_k$ is unaltered.

**Exercise 3**

Using $AV_k = V_{k+1}H_k$ for $k < \ell$ and the definitions of $\{\alpha_k, \beta_k, v_k\}$, show that the columns of $V_k$ are orthonormal ($k \leq \ell$).

**Exercise 4**

Show that the following subproblems are equivalent for defining how CG chooses $x_k$ when $A$ is positive definite:

1. minimize $\frac{1}{2} x_k^T A x_k - b^T x_k$ such that $x_k \in K_k$
2. minimize $\|r_k\|_{A^{-1}}$ such that $x_k \in K_k$
3. find $x_k$ such that $x_k \in K_k$ and $r_k \perp K_k$,

where $r_k = b - Ax_k$ and $\|w\|_{A^{-1}} = \sqrt{w^T A^{-1} w}$ for all $w \in \mathbb{R}^n$.

**Exercise 5** (extra credit)

Prove that $T_\ell$ is nonsingular if and only if $b \in \text{range}(A)$. Deduce that $\beta_{\ell+1} = 0$ is a “lucky breakdown” for CG, MINRES, and SYMMLQ if $b \in \text{range}(A)$.