The Lagrange Function for General Optimization and the Dual Problem

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Chapter 11.7-11.8, Chapter 3.1-3.5

Optimality Conditions for Unconstrained Optimization

Consider the unconstrained problem, where f is differentiable on R^n,

$$\text{(UO)} \qquad \begin{array}{ll} \min & f(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in R^n. \end{array}$$

Theorem 1 *Let* $\mathbf{x}$ *be a (local) minimizer of (UP) where* f *is continuously differentiable at* $\mathbf{x}$ *. Then*

$$\nabla f(\mathbf{x}) = \mathbf{0}.$$

These conditions are sufficient if $f(.)$ is a convex function of $\mathbf{x}$*. The one that meets the condition is called first-order KKT solution.*

A Nonlinear Optimization Example

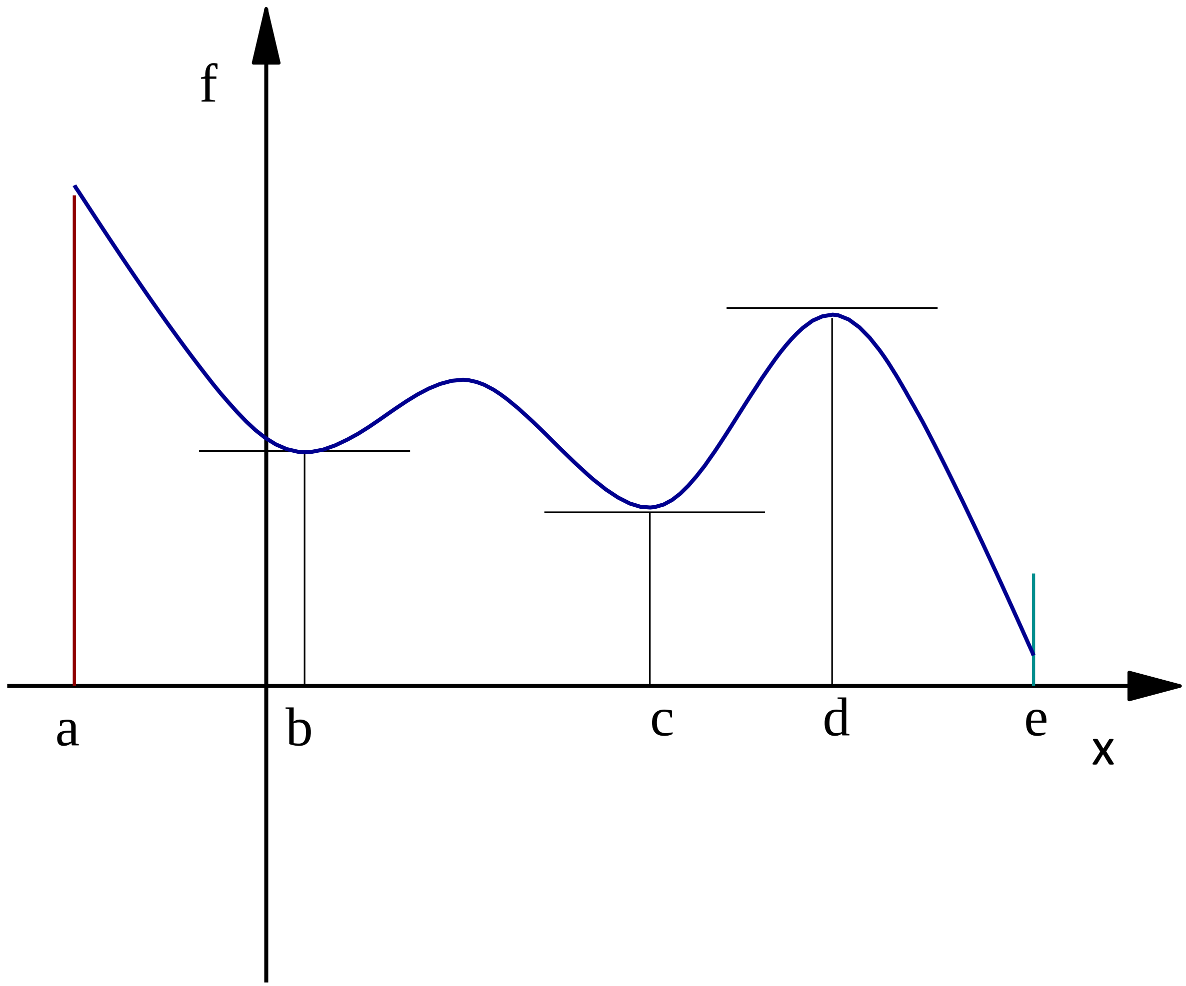


Figure : First-Order KKT solutions of one-variable function: b), c), d).

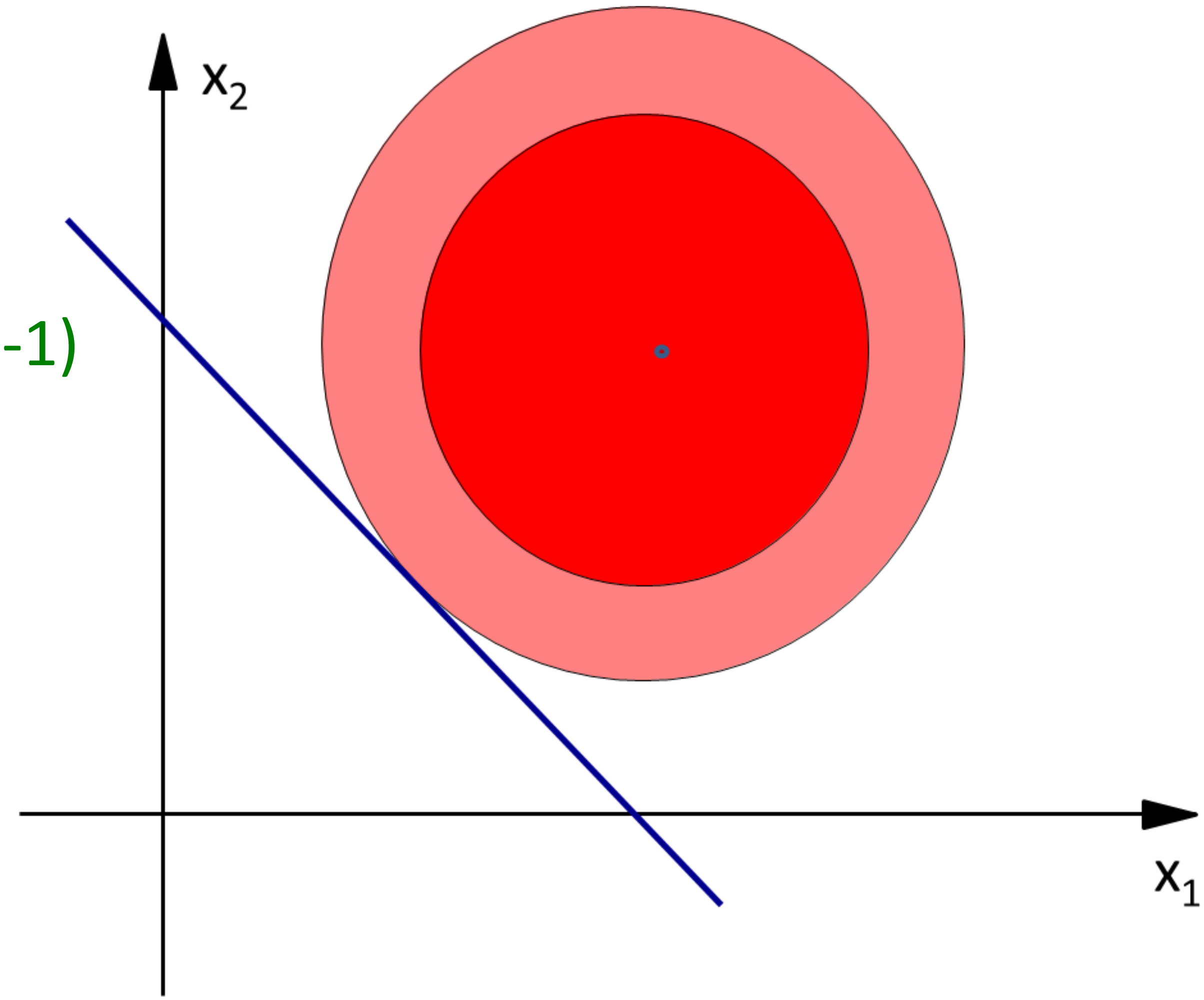
Optimality Condition for Constrained Optimization via the Lagrange Function

min $\quad (x_1 - 1)^2 + (x_2 - 1)^2$

s.t. $\quad x_1 + x_2 = 1 \qquad (y)$

$L(\mathbf{x},y)=(x_1 - 1)^2+(x_2 - 1)^2-y(x_1+x_2 -1)$

The key question is how to choose the penalty multiplier y such that the minimizer of the unconstrained Lagrangian problem yields the minimizer of the original problem, similar to the LP case discussed earlier.



How to find an accurate penalty weight y independently?

Penalty Principle: "Dual" Function from the Lagrangian

$$L(\boldsymbol{x},y)=(x_1-1)^2+(x_2-1)^2-y(x_1+x_2-1)$$

For any given and fixed y, the minimization of the Lagrangian is a unconstrained minimization problem so that the gradient of the Lagrangian must be a zero vector

$$\partial L(\boldsymbol{x},y)/\partial x_1 = 2x_1 - 2 - y = 0$$
$$\partial L(\boldsymbol{x},y)/\partial x_2 = 2x_2 - 2 - y = 0$$

Thus we must have $x_1 = 1 + y/2$ and $x_2 = 1 + y/2$

Substitute $\boldsymbol{x}$ by the expression of y, the minimal Lagrangian value becomes a function of y:

$$-y^2/2 - y$$

We call this minimal function of the multipliers

the Dual Function of the Lagrangian

Note that y=-1 is the maximizer of the dual function.

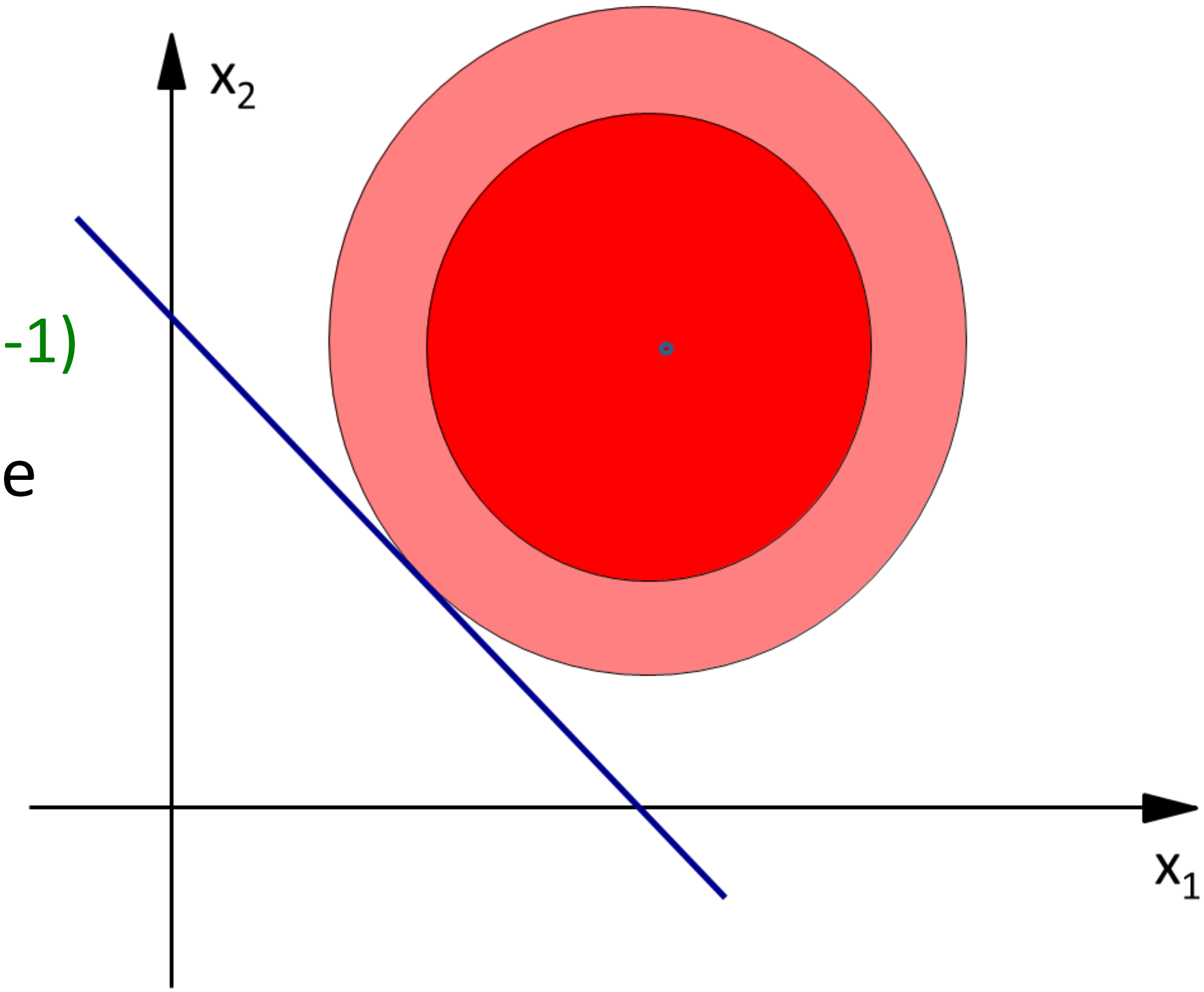
Now Solve the Unconstrained Lagrange Minimization

min $\quad (x_1 - 1)^2 + (x_2 - 1)^2$

s.t. $\quad x_1 + x_2 = 1 \qquad (y=-1)$

$L(\mathbf{x},y)=(x_1 - 1)^2+(x_2 - 1)^2+(x_1+x_2 -1)$

The gradient of the Lagrange function:



The Dual Function of General (Convex) Minimization

$$\min \quad f(\boldsymbol{x})$$
$$\text{s.t.} \quad c_i(\boldsymbol{x})\ (\le,=,\ge)\ 0,\ i=1,\dots,m$$
$$L(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x}) - \sum_i y_i c_i(\boldsymbol{x}),\ y_i\ (\le,\text{free},\ge) 0$$

- $f(\boldsymbol{x})$: convex function, $c_i(\boldsymbol{x})$: concave function of $\boldsymbol{x}$ for "≥" and convex function of $\boldsymbol{x}$ for "≤" ; and affine function *of* $\boldsymbol{x}$ *for "="*
- $L(\boldsymbol{x},\boldsymbol{y})$: would be a convex function of $\boldsymbol{x}$.
- Suppose for any given $\boldsymbol{y}$ (≤,free,≥) $\boldsymbol{0}$, define the dual function

$$\phi(\boldsymbol{y}) := \min_x L(\boldsymbol{x},\boldsymbol{y}) \ (\text{ or } \inf_x L(\boldsymbol{x},\boldsymbol{y})\ )$$

The Dual Problem of General (Convex) Minimization

$f^* := \min f(\boldsymbol{x})$

s.t. $c_i(\boldsymbol{x})\ (\le, =, \ge)\ 0,\ \forall i$

← Primal

$L(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x}) - \sum_i c_i(\boldsymbol{x}) y_i$

$\phi(\boldsymbol{y}) := \min_x\ L(\boldsymbol{x},\boldsymbol{y})$

$\phi^* := \max\ \phi(\boldsymbol{y})$

s.t. $\boldsymbol{y}\ (\le, \text{free}, \ge)\ 0,$

← Dual

Theorem

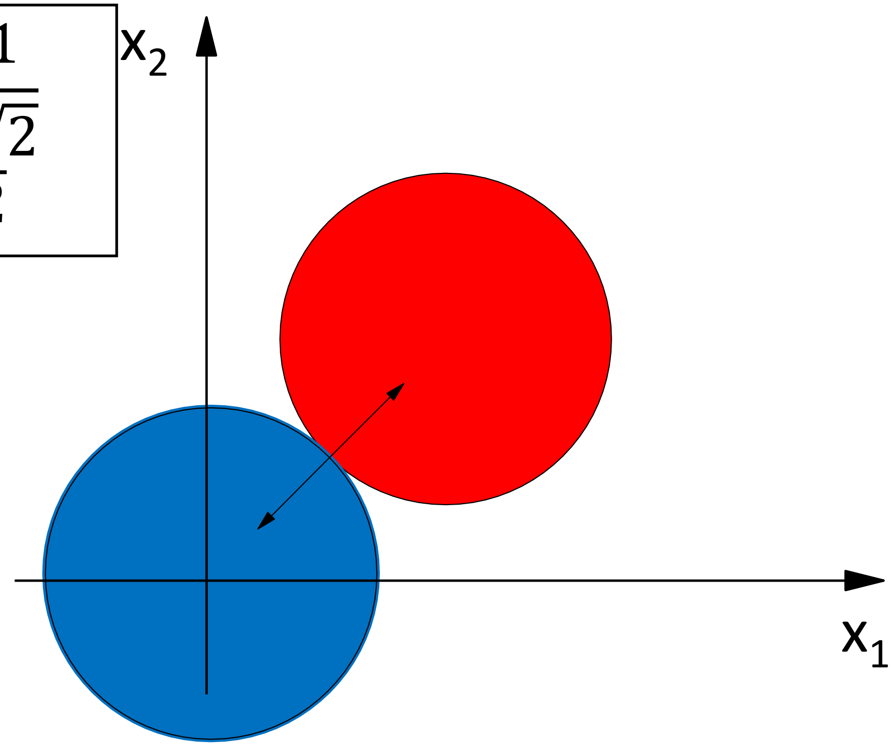
- $\phi(\boldsymbol{y})$ Is a concave function of $\boldsymbol{y}$ $(\le, \text{free}, \ge 0)$ ($\phi(\boldsymbol{y})$ can be $-\infty$)
- $\phi^* \le f^*$
- $\phi^* = f^*$ if the primal is a convex optimization (under mild technical assumptions), and $\nabla f^*(\boldsymbol{RHS}) = \boldsymbol{y}^*$ where $\boldsymbol{y}^*$ is the maximizer of the dual – Zero-Order Optimality Condition.

One can interpret the Lagrangian as a "game-value" where the ***x***-player minimizes it for given ***y***, and the ***y***-player maximizes it for given ***x***. The dual function is the anticipated function of the ***y*** decisions.

A Nonlinearly Constrained Optimization Example

$$\min \ (x_1-1)^2+(x_2-1)^2$$
$$\text{s.t.} \ -(x_1)^2-(x_2)^2 \ \ge -1$$

$$x_1^* = x_2^* = \frac{1}{\sqrt{2}}$$
$$f^* = 3-2\sqrt{2}$$



$$L(x_1,x_2,y)=$$
$$(x_1-1)^2+(x_2-1)^2$$
$$-y(1-(x_1)^2-(x_2)^2)\ ,$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+y} \\ \frac{1}{1+y} \end{pmatrix}$$

$$\varphi(y) = 2 - y - \frac{2}{1+y},$$
$$\max \ \varphi(y), \ \text{s.t.} \ y \ge 0$$
$$\varphi^* = 3-2\sqrt{2}$$
with $y^* = \sqrt{2}-1$

Dual

When RHS is reduced by 0.1?

General Rules to Construct the Dual

$\min f(\boldsymbol{x})$

$c_i(\boldsymbol{x})$ (≥,=,≤) 0, i=1,…,m (ODC)

Primal

Multiplier Sign Conditions (MSC)

y_i (≥,"free",≤) $0, i=1,\dots,m$

Constraints in the Dual

Lagrange Derivative Conditions (LDC)

$\partial L(\mathbf{x},\mathbf{y})/\partial x_j = 0$, for all $j=1,\dots,n$.

If no ***x*** in the equation, set it as an equality constraint in the dual; otherwise, express ***x*** in terms of ***y*** and replace ***x*** in the Lagrange function, which becomes the Dual objective.

Complementarity Slackness Condition (CSC)

$y_i c_i(\mathbf{x}) = 0$, for each inequality constraint i.

Not needed for construct Dual

The Dual of the LP Example I

$$
\begin{array}{llllllll}
\min & -x_1 & -2x_2 & & & & \\
\text{s.t.} & x_1 & & +x_3 & & & = 1 \\
& & x_2 & & +x_4 & & = 1 \\
& x_1 & +x_2 & & & +x_5 & = 1.5 \\
& x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
$$

For this example, recall that the (whole) Lagrangian would is

$L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{r})=-x_1-2x_2 - y_1(x_1+x_3-1) - y_2(x_2+x_4-1) - y_3(x_1+x_2+x_5-1.5) - \sum^5_{j=1} r_j x_j$

where the entries of $\boldsymbol{y}$ are the Lagrange multipliers associated with three equality constraints $A\boldsymbol{x}=\boldsymbol{b}$ and the entries of $\boldsymbol{r}(\geq \boldsymbol{0})$ are the multipliers associated with five inequality constraints $\boldsymbol{x} \geq \boldsymbol{0}$.

Reorganizing:

$$
L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{r})= (-1-y_1-y_3-r_1)x_1+(-2-y_2-y_3-r_2)x_2+(-y_1-r_3)x_3+(-y_2-r_4)x_4+(-y_3-r_5)x_5 + y_1+y_2+1.5y_3
$$

The Dual of the LP Example II

$L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{r}) = (-1-y_1-y_3-r_1)x_1+(-2-y_2-y_3-r_2)x_2+ (-y_1-r_3)x_3+(-y_2-r_4)x_4+(-y_3-r_5)x_5 + y_1+y_2+1.5y_3$

From the dual player "maximization" stand, if any coefficient of x_j in the Lagrangian is not zero, the primal or $\boldsymbol{x}$-player can choose $x_j=\infty$ or $-\infty$ to make the Lagrange function value down to $-\infty$.

Anticipate the behavior of the primal player, the optimal policy of the dual must choose $\boldsymbol{y}$ and $\boldsymbol{r}$ such that all coefficients to be zero and set them as constraints. Consequently, the Dual objective function becomes: $y_1+y_2+1.5y_3$, and the dual would be:

$$\max_{(y,r)} \; y_1 + y_2 + 1.5y_3$$

$$\text{s.t.} \quad (-1-y_1-y_3-r_1)=(-2-y_2-y_3-r_2)=(-y_1-r_3)=(-y_2-r_4)=(-y_3-r_5)=0,$$

$$r_j \geq 0, \; j=1,\ldots,5.$$

The Dual of LP Problem in Standard Equality Form

$$f^* := \min \quad c^T x$$
$$\text{s.t.} \quad Ax - \mathbf{b} = 0, \quad (y)$$
$$x \geq 0. \quad (r \geq 0)$$

$$\phi^* := \max \quad \phi(y, r)$$
$$\text{s.t.} \quad y \text{ free}, \; r \geq 0,$$

$$\phi^* := \max \; b^T y$$
$$\text{s.t.} \quad c - A^T y - r = 0, \; r \geq 0$$

Again $r = c - A^T y \in R^n$ called reduced cost vector or dual slacks vector.

$$L(x, y, r) = c^T x - y^T(Ax - b) - r^T x$$
$$= (c - A^T y - r)^T x + b^T y$$
$$\phi(y, r) := \min_x \; L(x, y, r)$$

Note that $\phi(y)$ is unbounded from below whenever

$$c - A^T y - r \neq 0$$

so that the dual would always enforce

$$c - A^T y - r = 0$$

The dual can be reduced as

$$\max \quad b^T y$$
$$\text{s.t.} \quad c - A^T y \geq 0$$

which is in the standard inequality form.

The Dual Function of (Concave) Maximization

$$\max \quad f(\boldsymbol{x})$$
$$\text{s.t.} \quad c_i(\boldsymbol{x})\ (\geq,=,\leq)\ 0,\ \text{I} = 1,\ldots,m$$

$$L(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x}) - \sum_i c_i(\boldsymbol{x}) y_i,\ y_i\ (\leq,\text{free},\geq) \boldsymbol{0}$$

- $f(\boldsymbol{x})$: concave function, $c_i(\boldsymbol{x})$: convex function of $\boldsymbol{x}$ for "≤" and concave function for "≥", and affine function for "=".
- $L(\boldsymbol{x},\boldsymbol{y})$: would be a concave function of $\boldsymbol{x}$.
- Suppose for any given $\boldsymbol{y}$ (≤,free,≥) $\boldsymbol{0}$, define the dual function

$$\phi(\boldsymbol{y}) := \max_x L(\boldsymbol{x},\boldsymbol{y}) \ (\text{ or } \sup_x L(\boldsymbol{x},\boldsymbol{y}))$$

The Dual Problem of (Concave) Maximization

$f^*:=\max f(\boldsymbol{x})$

s.t. $c_i(\boldsymbol{x})\ (\geq,=,\leq)\ 0,\ \forall\ i$

Primal

$L(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x}) - \sum_i c_i(\boldsymbol{x}) y_i$

$\phi(\boldsymbol{y}):=\max_{\boldsymbol{x}}\ L(\boldsymbol{x},\boldsymbol{y})$

$\phi^*:=\min\quad \phi(\boldsymbol{y})$

s.t. $\boldsymbol{y}\ (\leq,\text{free},\geq)\ 0,$

Dual

Theorem

- $\phi(\boldsymbol{y})$ Is a convex function of $\boldsymbol{y}\ (\leq,\text{free},\geq 0)$ ($\phi(\boldsymbol{y})$ can be ∞)
- $\phi^* \geq f^*$
- $\phi^* = f^*$ if the primal is a concave maximization (under mild technical assumptions); and $\nabla f^*(\boldsymbol{RHS})=\boldsymbol{y}^*$ where $\boldsymbol{y}^*$ is the minimizer of the dual.

The Dual of a Nonlinear Maximization Example

$$\begin{aligned} \max \quad & c_1 x_1 + c_2 x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1, \qquad \wedge\, y_1 : \text{free} \\ & (x_1)^2 + (x_2)^2 \le 1, \ \wedge\, y_2 \ge 0 \end{aligned}$$

Primal

$$\begin{aligned} L(x_1, x_2, y) = {} & c_1 x_1 + c_2 x_2 - y_1(x_1 + x_2 - 1) \\ & - y_2((x_1)^2 + (x_2)^2 - 1), \end{aligned}$$

$$\begin{pmatrix} c_1 - y_1 - 2y_2 x_1 \\ c_2 - y_1 - 2y_2 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\phi(y) = \frac{(c_1 - y_1)^2 + (c_2 - y_1)^2}{4y_2} + y_1 + y_2,$$

$$\min \ \phi(y), \ \text{s.t.} \ y_1 \text{ free}, y_2 \ge 0$$

Dual

The Dual of LP Problem in Standard Inequality Form

$$f^* := \max \ \boldsymbol{b}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} - \mathbf{c} \le \mathbf{0} \ (\boldsymbol{y} \ge 0)$$

$$L(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{b}^T\boldsymbol{x} - \boldsymbol{y}^T(A\boldsymbol{x} - \boldsymbol{c})$$
$$= (\boldsymbol{b} - A^T\boldsymbol{y})^T\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{y}$$
$$\phi(\boldsymbol{y}) := \max_{\boldsymbol{x}} \ L(\boldsymbol{x},\boldsymbol{y})$$

$$\phi^* := \min \ \phi(\boldsymbol{y})$$
$$\text{s.t.} \quad \boldsymbol{y} \ge \boldsymbol{0},$$

$$\phi^* := \min \ \boldsymbol{c}^T\boldsymbol{y}$$
$$\text{s.t.} \quad A^T\boldsymbol{y} - \boldsymbol{b} = \boldsymbol{0}, \ \boldsymbol{y} \ge \boldsymbol{0}$$

Note that $\phi(\boldsymbol{x})$ is unbounded from above whenever

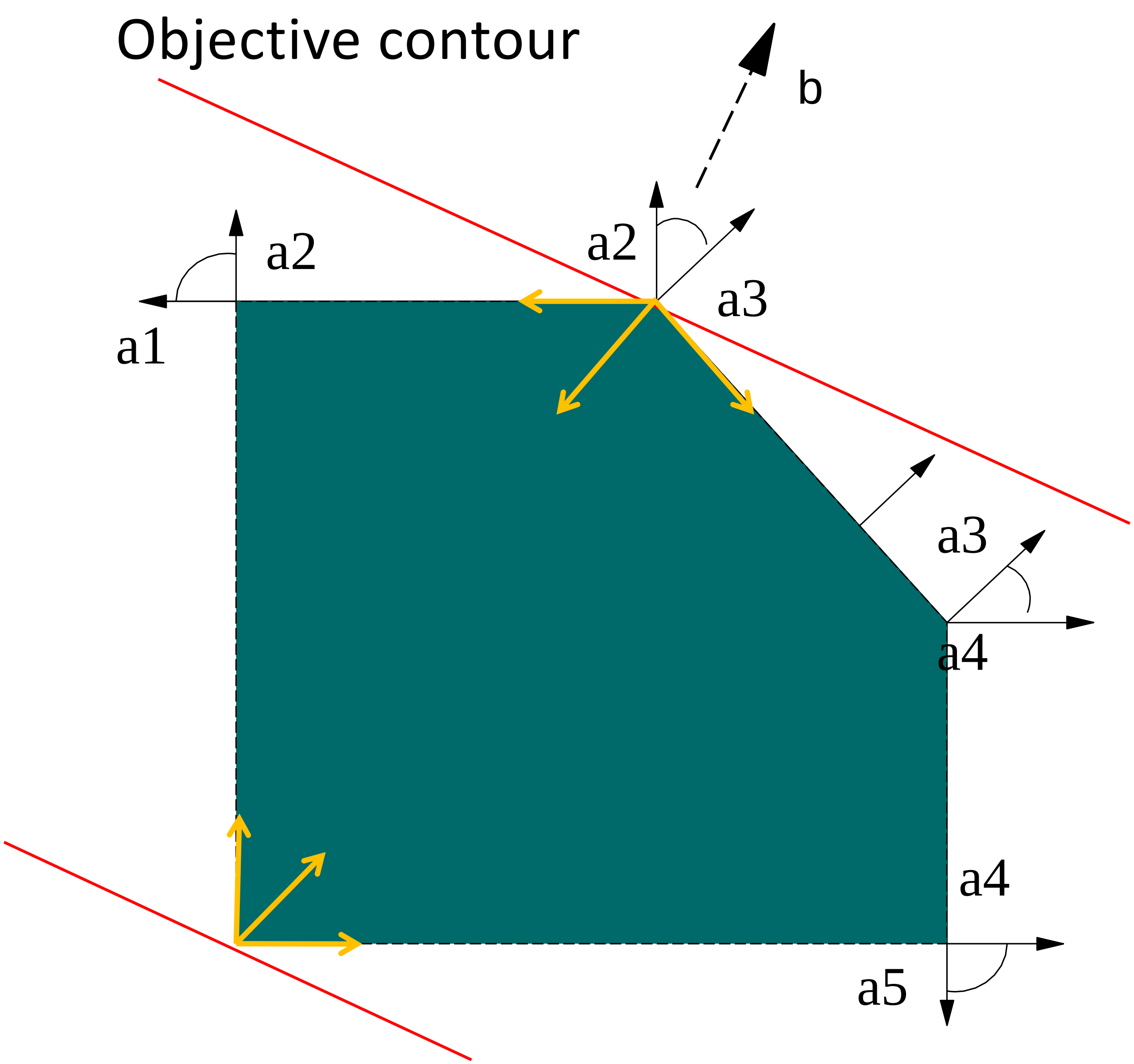
$$A^T\boldsymbol{y} - \boldsymbol{b} \neq \boldsymbol{0}$$

so that the dual would always enforce

$$A^T\boldsymbol{y} - \boldsymbol{b} = \boldsymbol{0}$$

and it is a LP in Standard Equality Form.

Geometric Interpretation of Dual Variables



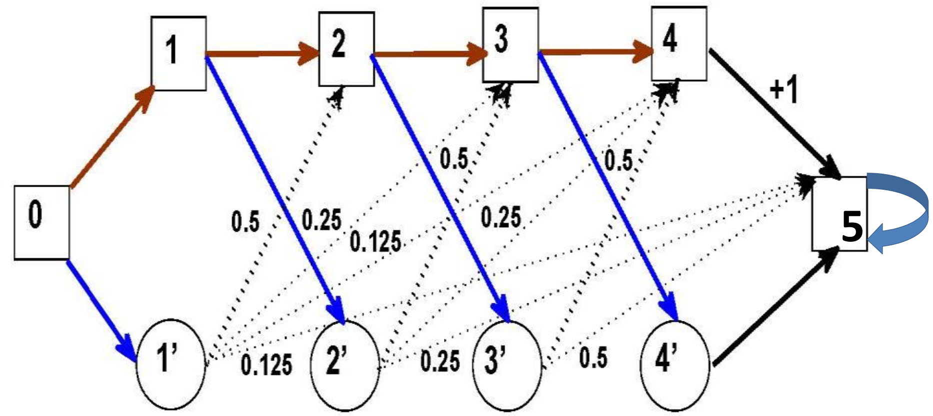
At the optimal corner, c must a **conic combination** of $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$, the two normal direction vectors of the intersection constraints. Or it has an obtuse angle with any (extreme) feasible directions.

Recall conic comb means there are **multipliers** $y_2 \geq 0$ and $y_3 \geq 0$, such as

$$\boldsymbol{b} = y_2 \boldsymbol{a_2} + y_3 \boldsymbol{a_3},$$

where all other multipliers are zeros.

Consider a Simplified MDP-RL Problem (Maze-Run)



$$\max\ y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

s.t.

$$y_5 \le 0 + \gamma y_5$$
$$y_4 \le 1 + \gamma y_5$$
$$y_3 \le 0 + \gamma y_4$$
$$y_3 \le 0 + \gamma y_5$$
$$y_2 \le 0 + \gamma y_3$$
$$y_2 \le 0 + \gamma(0.5y_4 + 0.5y_5)$$
$$y_1 \le 0 + \gamma y_2$$
$$y_1 \le 0 + \gamma(0.5y_3 + 0.25y_4 + 0.25y_5)$$
$$y_0 \le 0 + \gamma y_1$$
$$y_0 \le 0 + \gamma(0.5y_2 + 0.25y_3 + 0.125y_4 + 0.125y_5)$$

- y_i: expected overall cost if stating from State i.
- State 4 is a trap
- State 5 is the destination
- Each other state has two options: Go directly to the next state OR a short-cut go to other states with uncertainties

$$\mathbf{y^*_0 = y^*_1 = y^*_2 = y^*_3 = y^*_5 = 0}$$
$$\mathbf{y^*_4 = 1}$$

Physical Interpretation of the Maze-Run Dual

$\max\ y_0 + y_1 + y_2 + y_3 + y_4 + y_5$

s.t. $y_5 \le 0 + \gamma y_5 \quad (x_5)$

$y_4 \le 1 + \gamma y_5 \quad (x_4)$

$y_3 \le 0 + \gamma y_4 \quad (x_{3r})$

$y_3 \le 0 + \gamma y_5 \quad (x_{3b})$

$y_2 \le 0 + \gamma y_3 \quad (x_{2r})$

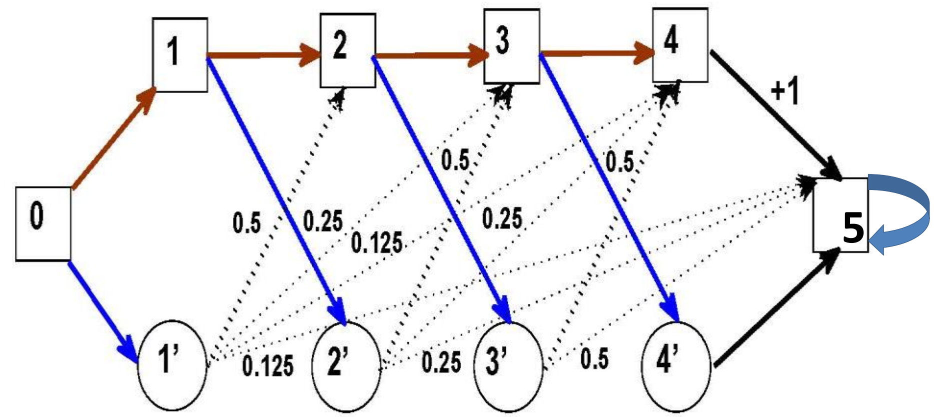
$y_2 \le 0 + \gamma(0.5y_4 + 0.5y_5) \quad (x_{2b})$

$y_1 \le 0 + \gamma y_2 \quad (x_{1r})$

$y_1 \le 0 + \gamma(0.5y_3 + 0.25y_4 + 0.25y_5) \quad (x_{1b})$

$y_0 \le 0 + \gamma y_1 \quad (x_{0r})$

$y_0 \le 0 + \gamma(0.5y_2 + 0.25y_3 + 0.125y_4 + 0.125y_5) \quad (x_{0b})$



x_j represents (discounted) how many expected times (frequency) actions j being taken in a policy.

The Dual of the Maze Example

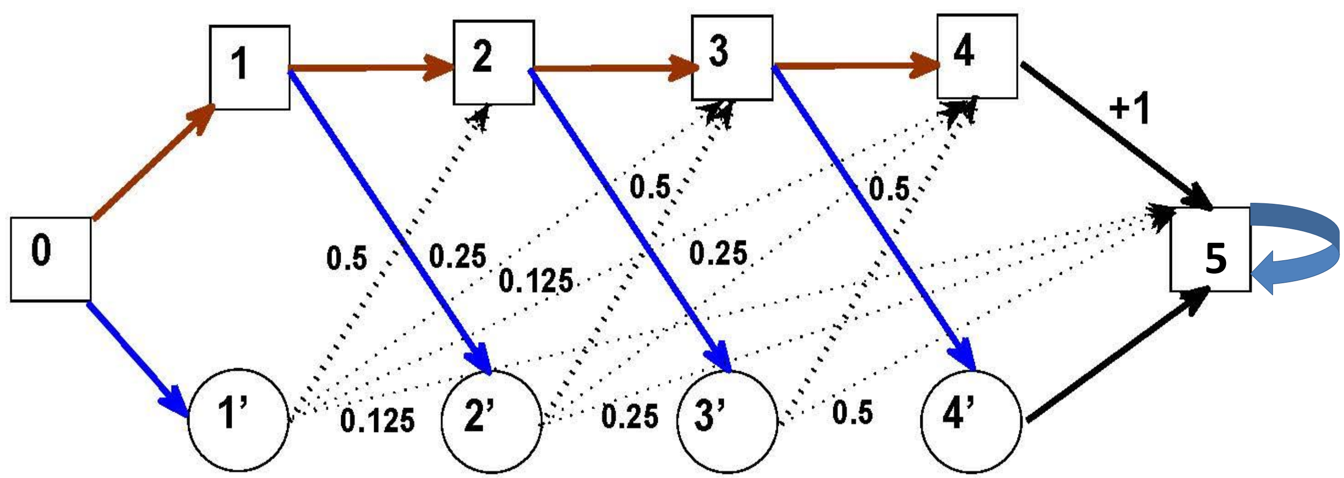
x:	(0r)	(0b)	(1r)	(1b)	(2r)	(2b)	(3r)	(3b)	(4)	(5)	b
c:	0	0	0	0	0	0	0	0	1	0	
(0)	1	1	0	0	0	0	0	0	0	0	1
(1)	$-\gamma$	0	1	1	0	0	0	0	0	0	1
(2)	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	0	0	1
(3)	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	1
(4)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	0	1
(5)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	0	$-\gamma$	$-\gamma$	$1-\gamma$	1

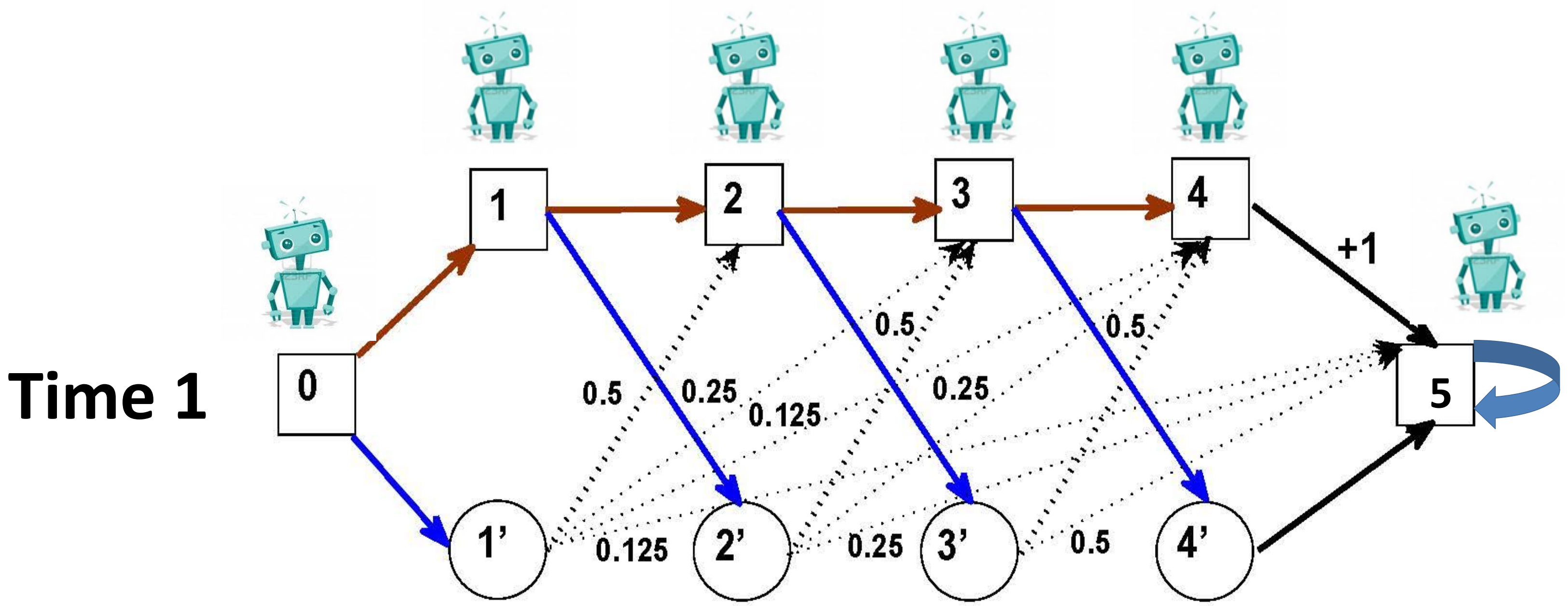
min $\boldsymbol{c}^T\boldsymbol{x}$
s.t. $A\boldsymbol{x} = \mathbf{e}$, $(\boldsymbol{y})$
$\boldsymbol{x} \geq \boldsymbol{0}$.

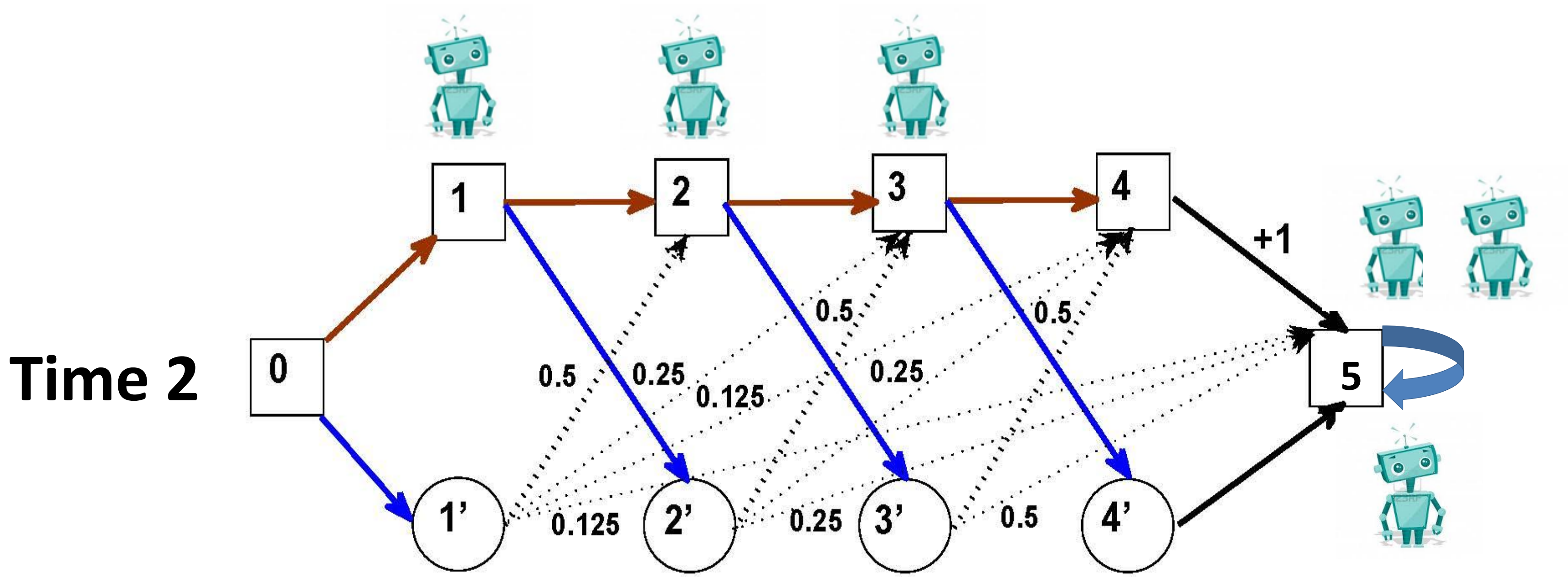
The optimal dual solution is

$$x^*_{0r} = 1,\ x^*_{1r} = 1+\gamma,\ x^*_{2r} = 1+\gamma+\gamma^2,\ x^*_{3b} = 1+\gamma+\gamma^2+\gamma^3,\ x^*_4 = 1,$$

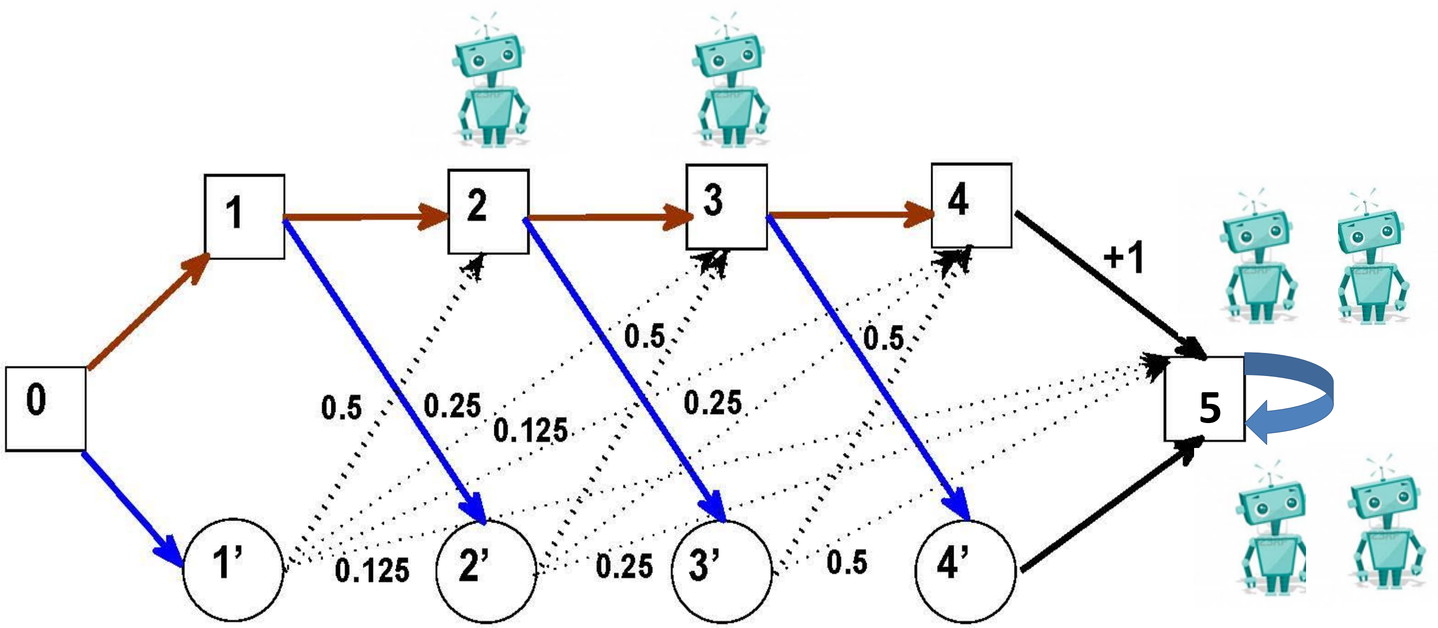
$$x^*_5 = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$




Time 1
0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5


Time 2
0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5

Time 3


0
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1'
2'
3'
4'
+1
0.5
0.25
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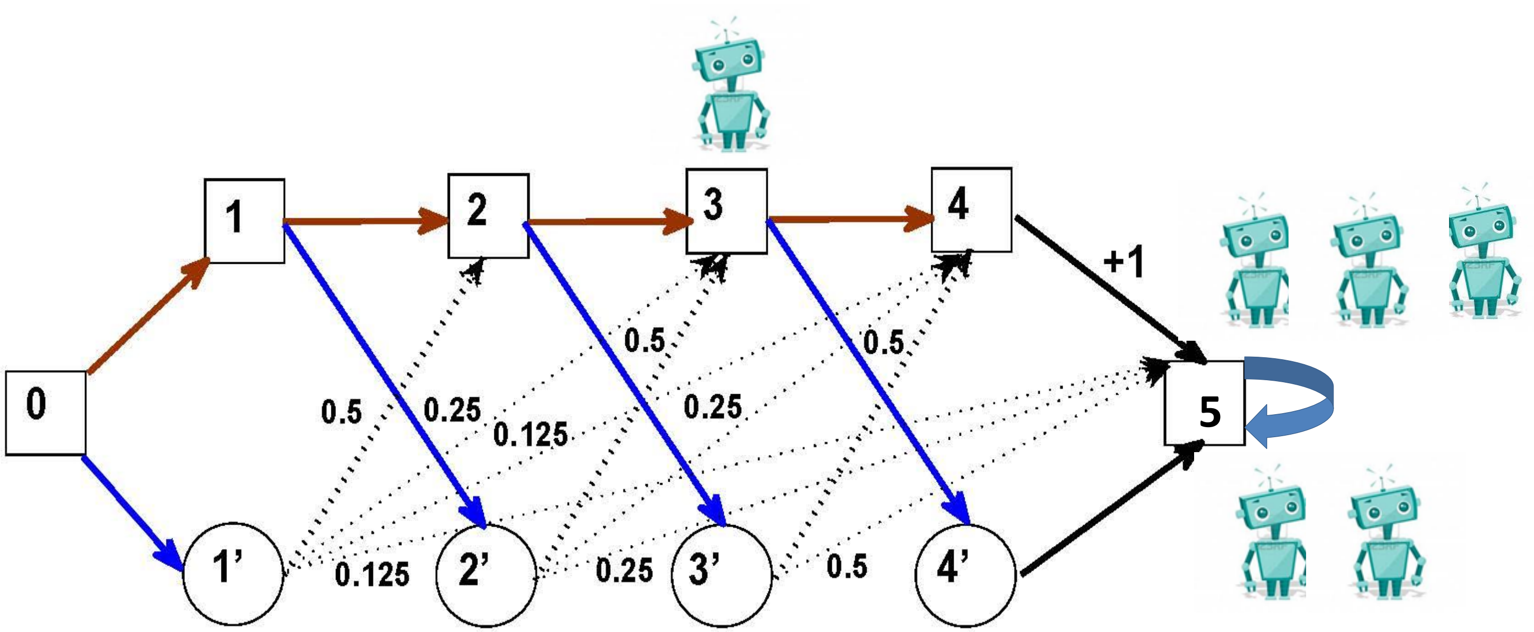
0.125
0.25
0.5
0.25
0.5

Time 4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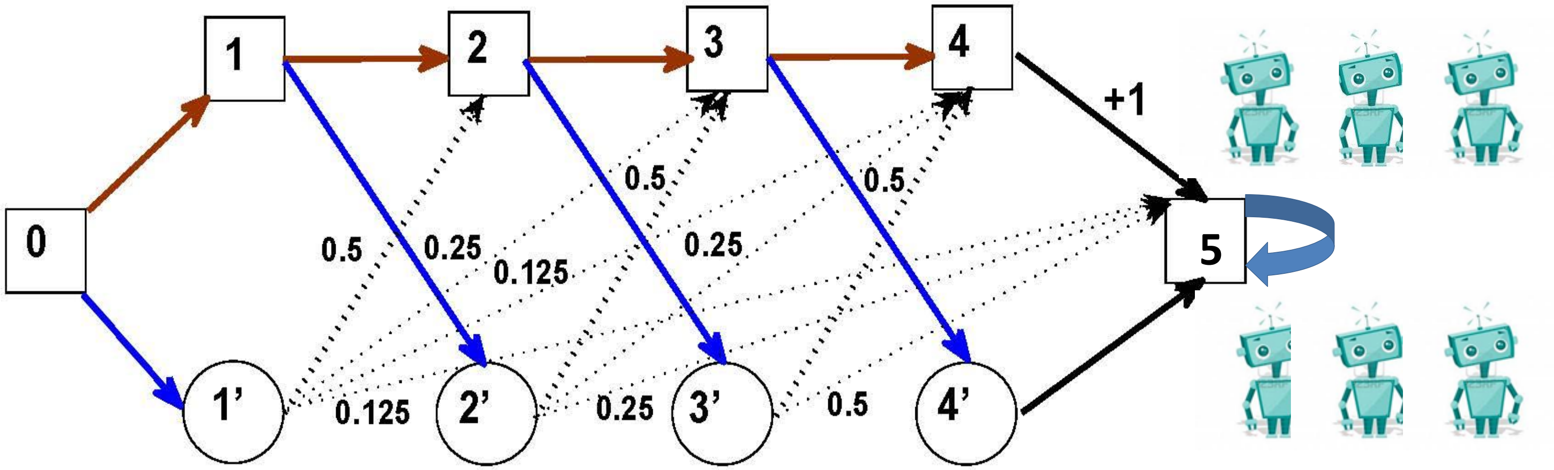
Time 5



Recall the optimal dual solution values are:

$$x^*_{0r} = 1,\ x^*_{1r} = 1 + \gamma,\ x^*_{2r} = 1 + \gamma + \gamma^2,\ x^*_{3b} = 1 + \gamma + \gamma^2 + \gamma^3,\ x^*_4 = 1,$$

$$x^*_5 = \frac{1 + 2\gamma + \gamma^2 + \gamma^3 + \gamma^4}{1 - \gamma}.$$

The Primal and Dual Problem of Optimization

- Every optimization problem is associated with another optimization problem called dual (the original problem is called primal).
- Every variable of the dual is the Lagrange multiplier associated with a constraint in the primal.
- The dual is max (min) if the primal is min (max)
- If the primal is a convex optimization problem, then the dual is also a convex optimization problem. Moreover, the two optimal objective values are equal (under mild technical assumptions).
- The optimal solution of the dual is the optimal Lagrange multiplier or shadow price vector of the primal.
- The above statements are also true if the constraint are nonlinear.

LP Duality Theorem

Theorem 1 *(Weak duality theorem) Let both primal feasible region F_p and dual feasible region F_d be non-empty. Then,*

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y} \qquad \textit{for all} \qquad \mathbf{x} \in F_p,\ \mathbf{y} \in F_d.$$

Proof: $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x} - (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T(\mathbf{c} - A^T\mathbf{y}) = \mathbf{x}^T\mathbf{r} \geq 0.$

This theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}$ the duality gap.

If the duality gap is zero, then $\mathbf{x}$ and $\mathbf{y}$ are optimal for the primal and dual, respectively! Is the reverse true?

LP Duality Theorem continued

Proved by the Simplex Method

Theorem 2 *(Strong duality theorem) Let both primal feasible region F_p and dual feasible region F_d be non-empty. Then, $\mathbf{x}^* \in F_p$ is optimal for (LP) and $\mathbf{y}^* \in F_d$ is optimal for (LD) if and only if the duality gap $\mathbf{c}^T\mathbf{x}^* - \mathbf{b}^T\mathbf{y}^* = 0$ (no need for technical assumptions).*

Corollary *If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.*

If one of (LP) or (LD) has no feasible solution, then the other is either unbounded or has no feasible solution. If one of (LP) or (LD) is unbounded then the other has no feasible solution.

Possible Outcome Combinations of Primal and Dual

Dual \ Primal	F-B	F-UB	IF
F-B	☺		☹
F-UB			☹
IF	☹	☹	☹

Only in Nonlinear Optimization

$$
\begin{array}{llcccl}
\min & -x_1 & - & x_2 & & \\
\text{s.t.} & x_1 & - & x_2 & = & 1 \\
 & -x_1 & + & x_2 & = & 1 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{llcccl}
\max & y_1 & + & y_2 & & \\
\text{s.t.} & y_1 & - & y_2 & \leq & -1 \\
 & -y_1 & + & y_2 & \leq & -1
\end{array}
$$