Optimization Duality Theory and Dual Interpretations

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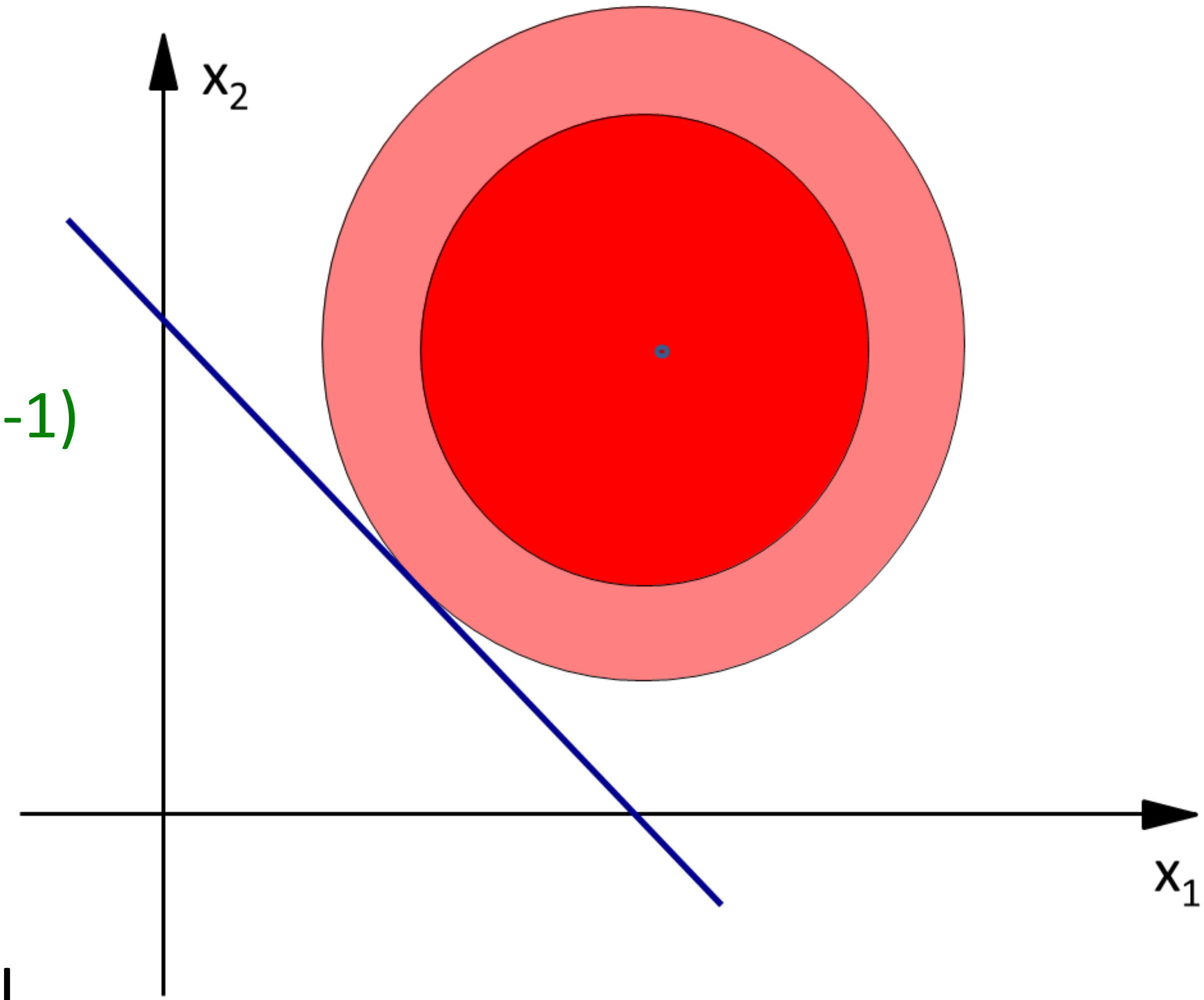
Chapter 11.7-11.8, Chapter 3.1-3.5

Recall the Constrained Quadratic Example

min $(x_1 - 1)^2 + (x_2 - 1)^2$

s.t. $x_1 + x_2 = 1 \quad (y)$

$L(\mathbf{x},y)=(x_1 - 1)^2+(x_2 - 1)^2-y(x_1+x_2 -1)$



The key question is how to choose the penalty multiplier *y* such that the minimizer of the Lagrangian is a minimizer of the original linear program.

The answer is *y=-1*

How to intelligently find such an accurate penalty weight?

Penalty Principle: Dual Function from the Lagrangian

$$L(\boldsymbol{x},y)=(x_1-1)^2+(x_2-1)^2-y(x_1+x_2-1)$$

For any given and fixed y, the minimization of the Lagrangian is a unconstrained minimization problem so that the gradient of the Lagrangian must be a zero vector

$$\partial L(\boldsymbol{x},y)/\partial x_1 = 2x_1 - 2 - y = 0$$
$$\partial L(\boldsymbol{x},y)/\partial x_2 = 2x_2 - 2 - y = 0$$

Thus we must have $x_1 = 1 + y/2$ and $x_2 = 1 + y/2$

Substitute $\boldsymbol{x}$ by the expression of y, the minimal Lagrangian becomes a function of y:

$$-y^2/2 - y$$

We call this minimal function of the multipliers

Dual Function of the Lagrangian

Note that y=-1 is the maximizer of the dual function

The Dual Function of (Convex) Minimization

$$\min \quad f(\boldsymbol{x})$$
$$\text{s.t.} \quad c_i(\boldsymbol{x}) \ (\le,=,\ge)\ 0\ , \ i = 1,\ldots,m$$
$$L(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x}) - \sum_i y_i\, c_i(\boldsymbol{x})\ ,\ y_i\ (\le, \text{free}, \ge) 0$$

- $f(\boldsymbol{x})$: convex function, $c_i(\boldsymbol{x})$: concave function of $\boldsymbol{x}$ for "≥" and convex function of $\boldsymbol{x}$ for "≤" ; and affine function *of* $\boldsymbol{x}$ *for "="*
- $L(\boldsymbol{x},\boldsymbol{y})$: would be a convex function of $\boldsymbol{x}$.
- Suppose for any given $\boldsymbol{y}$ (≤,free,≥) $\boldsymbol{0}$, define the dual function

$$\phi(\boldsymbol{y}) := \min_x L(\boldsymbol{x},\boldsymbol{y}) \ (\text{ or } \inf_x L(\boldsymbol{x},\boldsymbol{y})\ )$$

The Dual Problem of (Convex) Minimization

$f^* := \min f(\boldsymbol{x})$

s.t. $c_i(\boldsymbol{x})\ (\leq, =, \geq)\ 0,\ \forall\ i$

← Primal

$L(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x}) - \sum_i c_i(\boldsymbol{x}) y_i$

$\phi(\boldsymbol{y}) := \min_{\boldsymbol{x}} L(\boldsymbol{x},\boldsymbol{y})$

$\phi^* := \max \quad \phi(\boldsymbol{y})$

s.t. $\boldsymbol{y}\ (\leq, \text{free}, \geq)\ 0,$

← Dual

Theorem

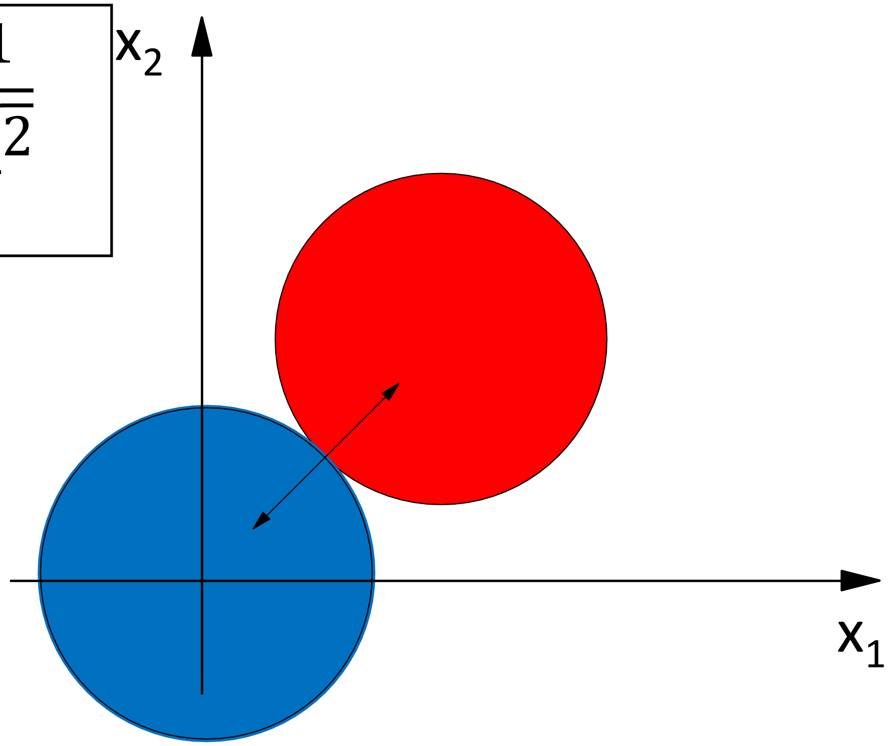
- $\phi(\boldsymbol{y})$ Is a concave function of $\boldsymbol{y}$ $(\leq, \text{free}, \geq 0)$ ($\phi(\boldsymbol{y})$ can be $-\infty$)
- $\phi^* \leq f^*$
- $\phi^* = f^*$ if the primal is a convex optimization (under mild technical assumptions), and $\nabla f^*(\boldsymbol{RHS}) = \boldsymbol{y}^*$ where $\boldsymbol{y}^*$ is the maximizer of the dual – Zero-Order Optimality Condition.

One can interpret the Lagrangian as a "game-value" where the ***x***-player minimizes it for given ***y***, and the ***y***-player maximizes it for given ***x***. The dual function is the anticipated function of the ***y*** decisions.

A Nonlinearly Constrained Optimization Example

$$\min \ (x_1-1)^2+(x_2-1)^2$$
$$\text{s.t.} \ -(x_1)^2-(x_2)^2 \ \geq -1$$

$$x_1^* = x_2^* = \frac{1}{\sqrt{2}}$$
$$f^* = 3-2\sqrt{2}$$



$$L(x_1,x_2,y)=$$
$$(x_1-1)^2+(x_2-1)^2$$
$$-y(1-(x_1)^2-(x_2)^2) \ ,$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+y} \\ \frac{1}{1+y} \end{pmatrix}$$

$$\varphi(y) = 2-y-\frac{2}{1+y},$$
$$\max \ \varphi(y), \text{ s.t. } y \geq 0$$
$$\varphi^* = 3-2\sqrt{2}$$
with $y^* = \sqrt{2}-1$

Dual

When RHS is reduced by 0.1?

General Rules to Construct the Dual

min $f(\boldsymbol{x})$

$c_i(\boldsymbol{x})$ (≥,=,≤) 0, i=1,…,m (ODC)

← **Primal**

Multiplier Sign Conditions (MSC)

y_i (≥,"free",≤) 0, $i=1,\ldots,m$

← **Constraints in the Dual**

Lagrange Derivative Conditions (LDC)

$\partial L(\mathbf{x},\mathbf{y})/\partial x_j = 0$, for all $j=1,\ldots,n$.

← If no $\boldsymbol{x}$ in the equation, set it as an equality constraint in the dual; otherwise, express $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ and replace $\boldsymbol{x}$ in the Lagrange function, which becomes the Dual objective.

Complementarity Slackness Condition (CSC)

$y_i c_i(\mathbf{x}) = 0$, for each inequality constraint i. ✗

Not needed for construct Dual

The Dual of the LP Example I

$$\begin{array}{llllllll} \min & -x_1 & -2x_2 & & & & \\ \text{s.t.} & x_1 & & +x_3 & & & = 1 \\ & & x_2 & & +x_4 & & = 1 \\ & x_1 & +x_2 & & & +x_5 & = 1.5 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0. \end{array}$$

For this example, the Lagrangian would be

$L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{r}) = -x_1 - 2x_2 - y_1(x_1 + x_3 - 1) - y_2(x_2 + x_4 - 1) - y_3(x_1 + x_2 + x_5 - 1.5) - \sum_{j=1}^{5} r_j x_j$

where the entries of $\boldsymbol{y}$ are the Lagrange multipliers associated with three equality constraints $A\boldsymbol{x}=\boldsymbol{b}$ and the entries of $\boldsymbol{r}(\geq \boldsymbol{0})$ are the multipliers associated with five inequality constraints $\boldsymbol{x} \geq \boldsymbol{0}$.

Reorganizing:

$L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{r}) = (-1-y_1-y_3-r_1)x_1 + (-2-y_2-y_3-r_2)x_2 + (-y_1-r_3)x_3 + (-y_2-r_4)x_4 + (-y_3-r_5)x_5$
$+ y_1 + y_2 + 1.5y_3$

The Dual of the LP Example II

The dual would be

$\max_{(y,r)}\ y_1 + y_2 + 1.5y_3$

s.t. $(-1-y_1-y_3-r_1)=(-2-y_2-y_3-r_2)=(-y_1-r_3)=(-y_2-r_4)=(-y_3-r_5)=0,$

$r_j \geq 0,\ j=1,\ldots,5.$

which can be simplified as

$\max_{y}\ y_1 + y_2 + 1.5y_3$

s.t. $-1-y_1-y_3\geq 0,\ -2-y_2-y_3\geq 0,\ -y_1\geq 0,\ -y_2\geq 0,\ -y_3\geq 0.$

From the dual stand, if any coefficient of x_j in the Lagrangian

$L(\mathbf{x},\mathbf{y},\mathbf{r})= (-1-y_1-y_3-r_1)x_1+(-2-y_2-y_3-r_2)x_2+(-y_1-r_3)x_3+(-y_2-r_4)x_4+(-y_3-r_5)x_5$
$+ y_1+y_2+1.5y_3$

is not zero, the primal or ***x***-player can choose $x_j=\infty$ or $-\infty$ to make the game-value down to $-\infty$.

Anticipate the behavior of the primal player, the optimal policy of the dual must choose ***y*** and ***r*** such that all coefficients to be zero, and the Dual objective function becomes: $y_1+y_2+1.5y_3$

The Dual of LP Problem in Standard Equality Form

$$f^* := \min \; \mathbf{c}^T\mathbf{x}$$
$$\text{s.t.} \; A\mathbf{x} - \mathbf{b} = 0, \; (\mathbf{y})$$
$$\mathbf{x} \geq 0. \; (\mathbf{r} \geq 0)$$

$$L(\mathbf{x},\mathbf{y},\mathbf{r}) = \mathbf{c}^T\mathbf{x} - \mathbf{y}^T(A\mathbf{x} - \mathbf{b}) - \mathbf{r}^T\mathbf{x}$$
$$= (\mathbf{c} - A^T\mathbf{y} - \mathbf{r})^T\mathbf{x} + \mathbf{b}^T\mathbf{y}$$
$$\phi(\mathbf{y},\mathbf{r}) := \min_x \; L(\mathbf{x},\mathbf{y},\mathbf{r})$$

$$\phi^* := \max \; \phi(\mathbf{y},\mathbf{r})$$
$$\text{s.t.} \; \mathbf{y} \text{ free}, \; \mathbf{r} \geq \mathbf{0},$$

$$\phi^* := \max \; \mathbf{b}^T\mathbf{y}$$
$$\text{s.t.} \; \mathbf{c} - A^T\mathbf{y} - \mathbf{r} = \mathbf{0}, \; \mathbf{r} \geq \mathbf{0}$$

Again $\mathbf{r} = \mathbf{c} - A^T\mathbf{y} \in R^n$ called reduced cost vector or dual slacks vector.

Note that $\phi(\mathbf{y})$ is unbounded from below whenever

$$\mathbf{c} - A^T\mathbf{y} - \mathbf{r} \neq \mathbf{0}$$

so that the dual would always enforce

$$\mathbf{c} - A^T\mathbf{y} - \mathbf{r} = \mathbf{0}$$

The dual can be reduced as

$$\max \; \mathbf{b}^T\mathbf{y}$$
$$\text{s.t.} \; \mathbf{c} - A^T\mathbf{y} \geq \mathbf{0}$$

which is in the standard inequality form.

The Dual Function of (Concave) Maximization

$$\max \quad f(\boldsymbol{x})$$
$$\text{s.t.} \quad c_i(\boldsymbol{x}) \ (\geq,=,\leq)\ 0,\ \text{I} = 1,\ldots,m$$

$$L(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x}) - \sum_i c_i(\boldsymbol{x}) y_i,\ y_i\ (\leq, \text{free}, \geq)\ \boldsymbol{0}$$

- $f(\boldsymbol{x})$: concave function, $c_i(\boldsymbol{x})$: convex function of $\boldsymbol{x}$ for "≤" and concave function for "≥", and affine function for "=".
- $L(\boldsymbol{x},\boldsymbol{y})$: would be a concave function of $\boldsymbol{x}$.
- Suppose for any given $\boldsymbol{y}$ (≤,free,≥) $\boldsymbol{0}$, define the dual function

$$\phi(\boldsymbol{y}) := \max_x L(\boldsymbol{x},\boldsymbol{y}) \ (\text{ or } \sup_x L(\boldsymbol{x},\boldsymbol{y}))$$

The Dual Problem of (Concave) Maximization

f^*:=max $f(\boldsymbol{x})$
s.t. $c_i(\boldsymbol{x})$ (≥,=,≤) 0, ∀ i

Primal

$L(\boldsymbol{x},\boldsymbol{y})$:=f($\boldsymbol{x}$)- $\sum_i c_i(\boldsymbol{x}) y_i$

$\phi(\boldsymbol{y})$:=max$_{\boldsymbol{x}}$ $L(\boldsymbol{x},\boldsymbol{y})$

ϕ^*:=min $\phi(\boldsymbol{y})$
s.t. $\boldsymbol{y}$ (≤,free,≥) 0,

Dual

Theorem

- $\phi(\boldsymbol{y})$ Is a convex function of $\boldsymbol{y}$ (≤,free,≥ 0) ($\phi(\boldsymbol{y})$ can be ∞)
- $\phi^* \geq f^*$
- $\phi^* = f^*$ if the primal is a concave maximization (under mild technical assumptions); and $\nabla f^*(\boldsymbol{RHS})=\boldsymbol{y}^*$ where $\boldsymbol{y}^*$ is the minimizer of the dual.

The Dual of a Nonlinear Maximization Example

$$\begin{array}{lll} \max & c_1x_1 + c_2x_2 & \\ \text{s.t.} & x_1 + x_2 = 1, & \wedge\, y_1 : \text{free} \\ & (x_1)^2 + (x_2)^2 \le 1, & \wedge\, y_2 \ge 0 \end{array}$$

Primal

$$L(x_1, x_2, y) = c_1x_1 + c_2x_2 - y_1(x_1 + x_2 - 1) - y_2((x_1)^2 + (x_2)^2 - 1),$$

$$\begin{pmatrix} c_1 - y_1 - 2y_2x_1 \\ c_2 - y_1 - 2y_2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\phi(y) = \frac{(c_1 - y_1)^2 + (c_2 - y_1)^2}{4y_2} + y_1 + y_2,$$

$$\min\ \phi(y),\ \text{s.t.}\ y_1\ \text{free}, y_2 \ge 0$$

Dual

The Dual of LP Problem in Standard Inequality Form

$$f^* := \max \ \mathbf{b}^T\mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} - \mathbf{c} \leq \mathbf{0} \ (\mathbf{y} \geq 0)$$

$$L(\mathbf{x},\mathbf{y}) = \mathbf{b}^T\mathbf{x} - \mathbf{y}^T(A\mathbf{x} - \mathbf{c})$$
$$= (\mathbf{b} - A^T\mathbf{y})^T\mathbf{x} + \mathbf{c}^T\mathbf{y}$$
$$\phi(\mathbf{y}) := \max_{\mathbf{x}} \ L(\mathbf{x},\mathbf{y})$$

$$\phi^* := \min \quad \phi(\mathbf{y})$$
$$\text{s.t.} \quad \mathbf{y} \geq \mathbf{0},$$

Note that $\phi(\mathbf{x})$ is unbounded from above whenever

$$A^T\mathbf{y} - \mathbf{b} \neq \mathbf{0}$$

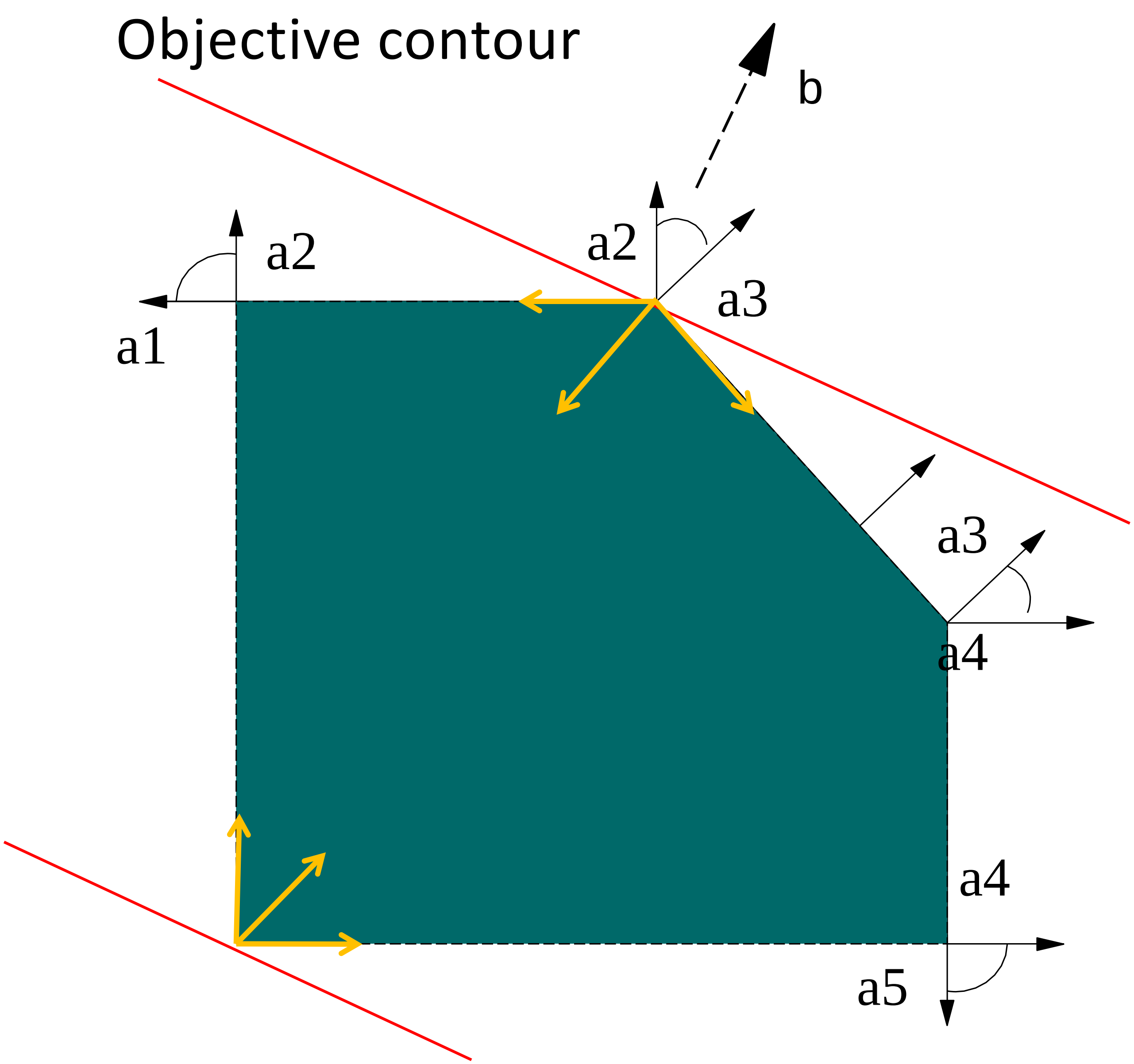
so that the dual would always enforce

$$A^T\mathbf{y} - \mathbf{b} = \mathbf{0}$$

and it is a LP in Standard Equality Form.

$$\phi^* := \min \ \mathbf{c}^T\mathbf{y}$$
$$\text{s.t.} \quad A^T\mathbf{y} - \mathbf{b} = \mathbf{0}, \ \mathbf{y} \geq \mathbf{0}$$

Geometric Interpretation of Dual Variables



At the optimal corner, c must a **conic combination** of $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$, the two normal direction vectors of the intersection constraints. Or it has an obtuse angle with any (extreme) feasible directions.

Recall conic comb means there are **multipliers** $y_2 \geq 0$ and $y_3 \geq 0$, such as

$$\boldsymbol{b} = y_2 \boldsymbol{a_2} + y_3 \boldsymbol{a_3},$$

where all other multipliers are zeros.

Consider a Simplified MDP-RL Problem (Maze-Run)

$$\max \; y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

$$\text{s.t.} \quad y_5 \le 0 + \gamma y_5$$

$$y_4 \le 1 + \gamma y_5$$

$$y_3 \le 0 + \gamma y_4$$

$$y_3 \le 0 + \gamma y_5$$

$$y_2 \le 0 + \gamma y_3$$

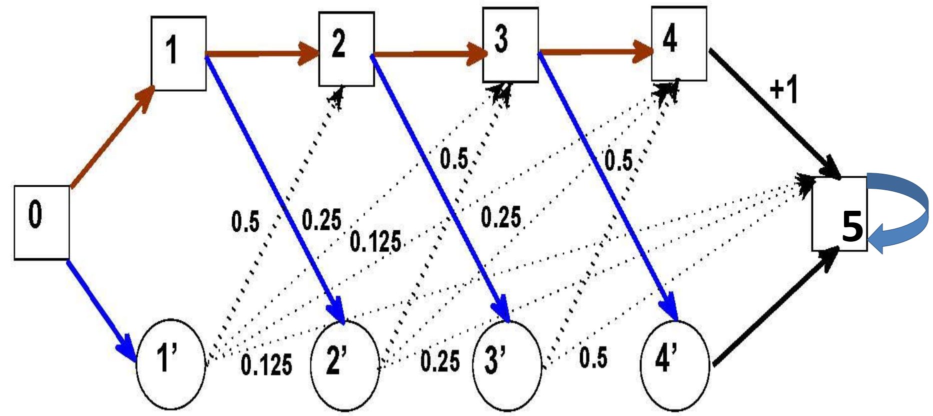
$$y_2 \le 0 + \gamma(0.5 y_4 + 0.5 y_5)$$

$$y_1 \le 0 + \gamma y_2$$

$$y_1 \le 0 + \gamma(0.5 y_3 + 0.25 y_4 + 0.25 y_5)$$

$$y_0 \le 0 + \gamma y_1$$

$$y_0 \le 0 + \gamma(0.5 y_2 + 0.25 y_3 + 0.125 y_4 + 0.125 y_5)$$



- y_i: expected overall cost if stating from State i.
- State 4 is a trap
- State 5 is the destination
- Each other state has two options: Go directly to the next state OR a short-cut go to other states with uncertainties

$$\mathbf{y^*_0 = y^*_1 = y^*_2 = y^*_3 = y^*_5 = 0}$$

$$\mathbf{y^*_4 = 1}$$

Physical Interpretation of the Maze-Run Dual

$$\max\ y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

s.t.

$y_5 \le 0 + \gamma y_5 \quad (x_5)$

$y_4 \le 1 + \gamma y_5 \quad (x_4)$

$y_3 \le 0 + \gamma y_4 \quad (x_{3r})$

$y_3 \le 0 + \gamma y_5 \quad (x_{3b})$

$y_2 \le 0 + \gamma y_3 \quad (x_{2r})$

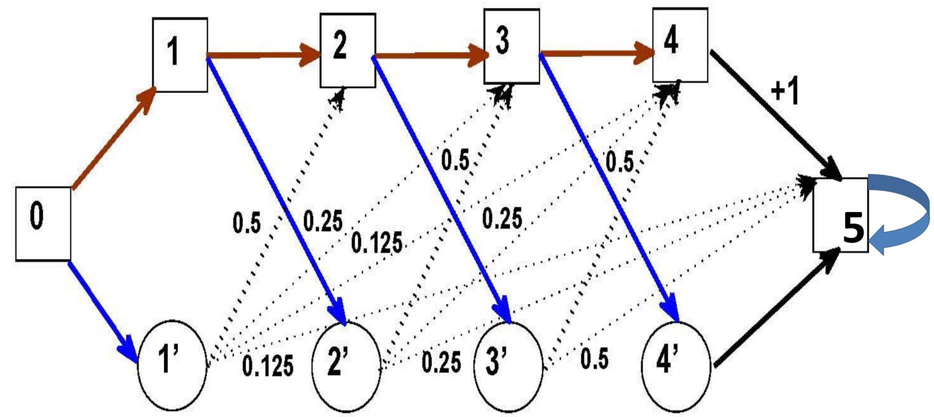
$y_2 \le 0 + \gamma(0.5y_4 + 0.5y_5) \quad (x_{2b})$

$y_1 \le 0 + \gamma y_2 \quad (x_{1r})$

$y_1 \le 0 + \gamma(0.5y_3 + 0.25y_4 + 0.25y_5) \quad (x_{1b})$

$y_0 \le 0 + \gamma y_1 \quad (x_{0r})$

$y_0 \le 0 + \gamma(0.5y_2 + 0.25y_3 + 0.125y_4 + 0.125y_5) \quad (x_{0b})$



x_j represents (discounted) how many expected times (frequency) actions j being taken in a policy.

The Dual of the Maze Example

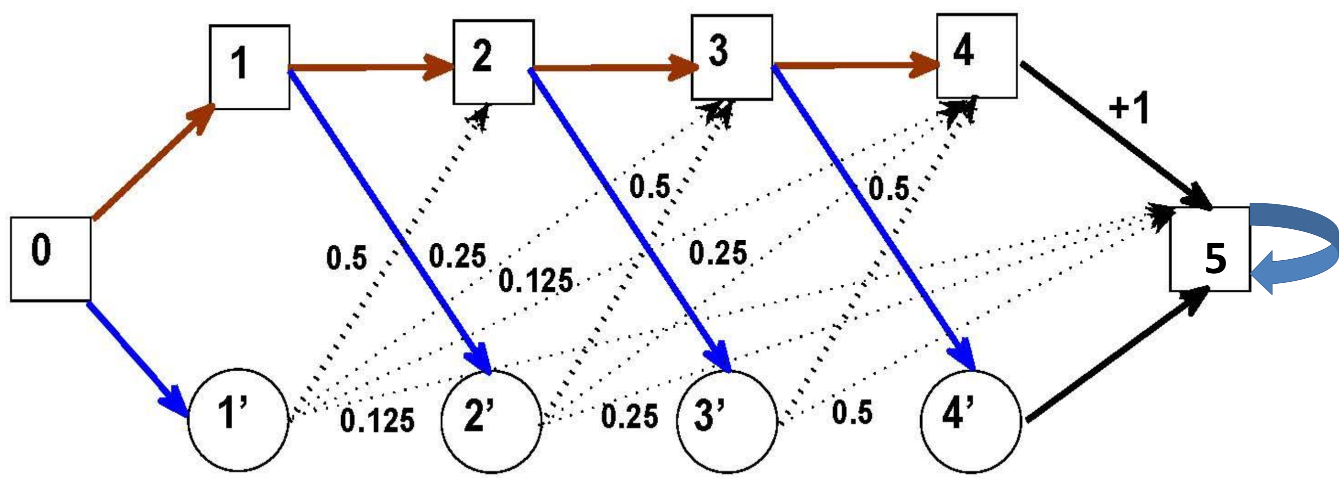
x:	$(0r)$	$(0b)$	$(1r)$	$(1b)$	$(2r)$	$(2b)$	$(3r)$	$(3b)$	(4)	(5)	b
c:	0	0	0	0	0	0	0	0	1	0	
(0)	1	1	0	0	0	0	0	0	0	0	1
(1)	$-\gamma$	0	1	1	0	0	0	0	0	0	1
(2)	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	0	0	1
(3)	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	1
(4)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	0	1
(5)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	0	$-\gamma$	$-\gamma$	$1-\gamma$	1

$$\begin{aligned} \min\ & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t. } & A\boldsymbol{x} = \mathbf{e}, \ (\boldsymbol{y}) \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

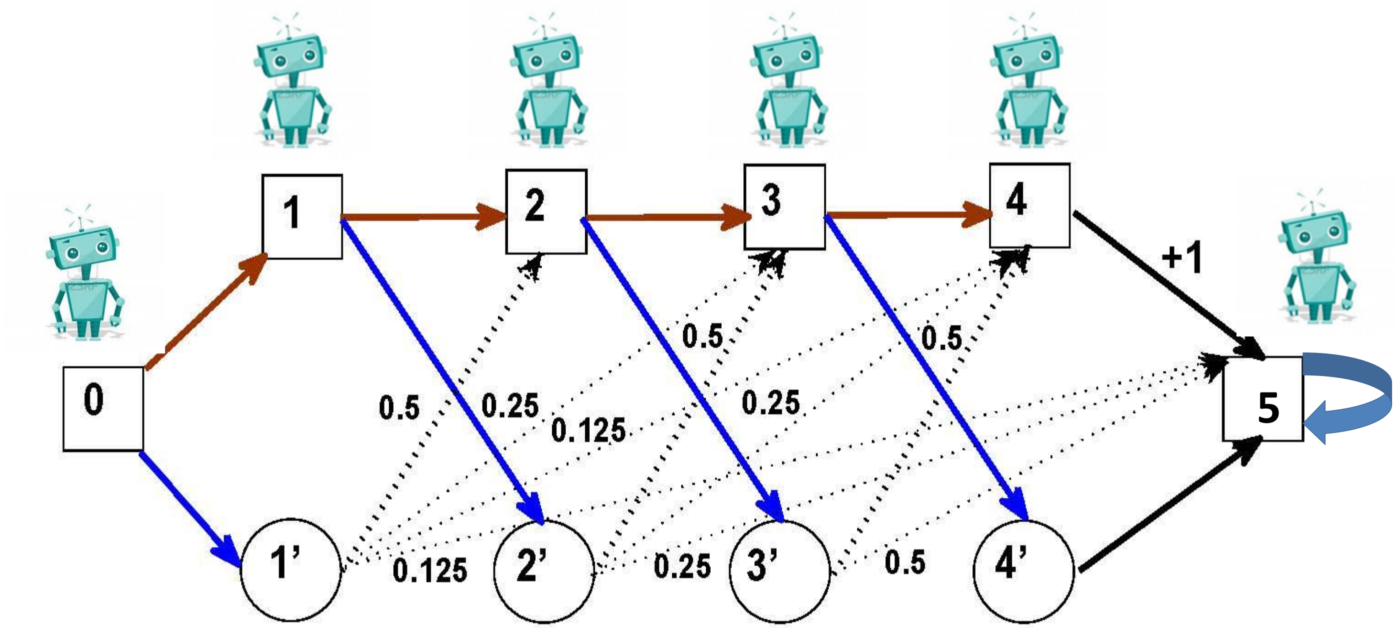
The optimal dual solution is

$$x^*_{0r} = 1,\ x^*_{1r} = 1+\gamma,\ x^*_{2r} = 1+\gamma+\gamma^2,\ x^*_{3b} = 1+\gamma+\gamma^2+\gamma^3,\ x^*_4 = 1,$$

$$x^*_5 = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$



Time 1



0
1
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1'
2'
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4'
+1
0.5
0.25
0.125
0.5
0.25
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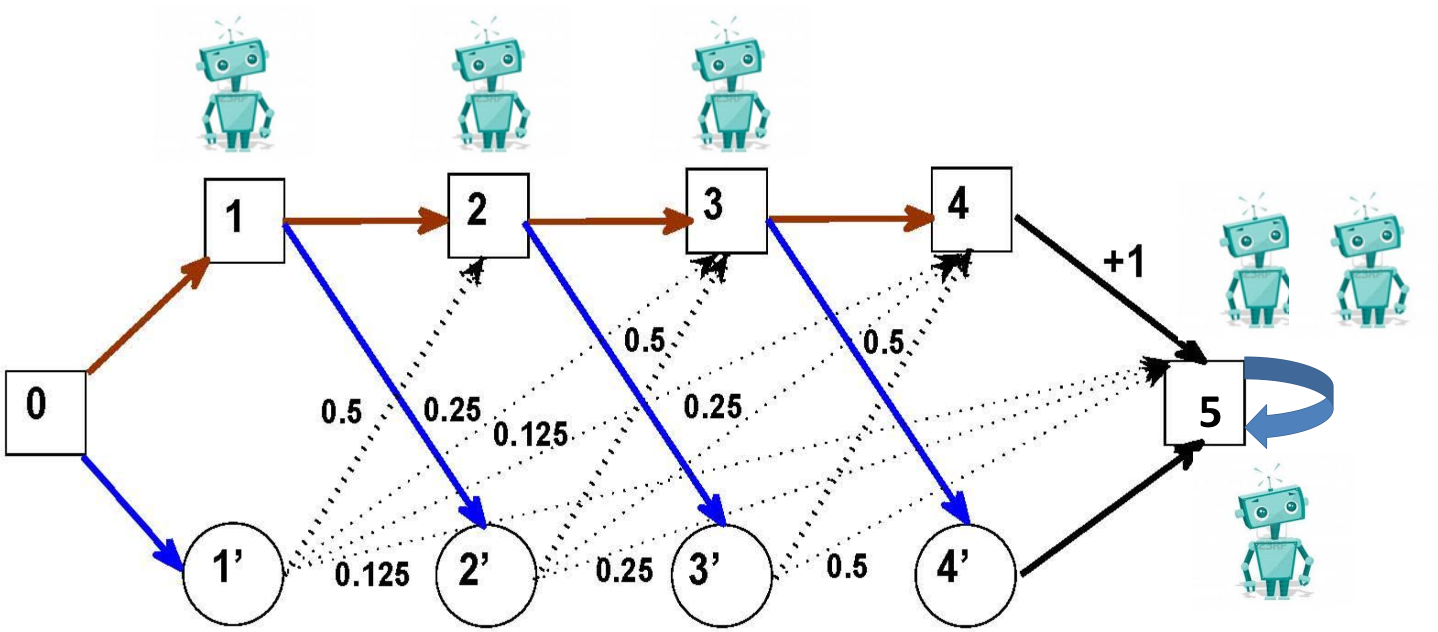
0.5

Time 2



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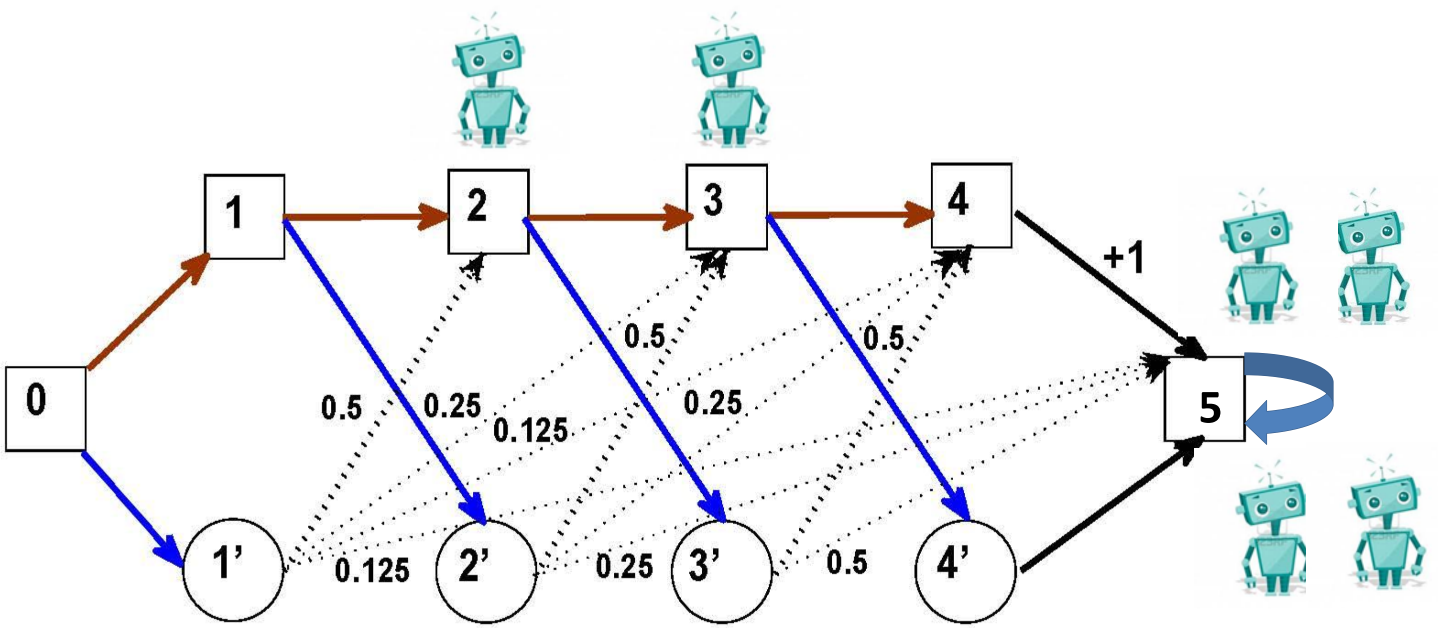
0.5

Time 3



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+1
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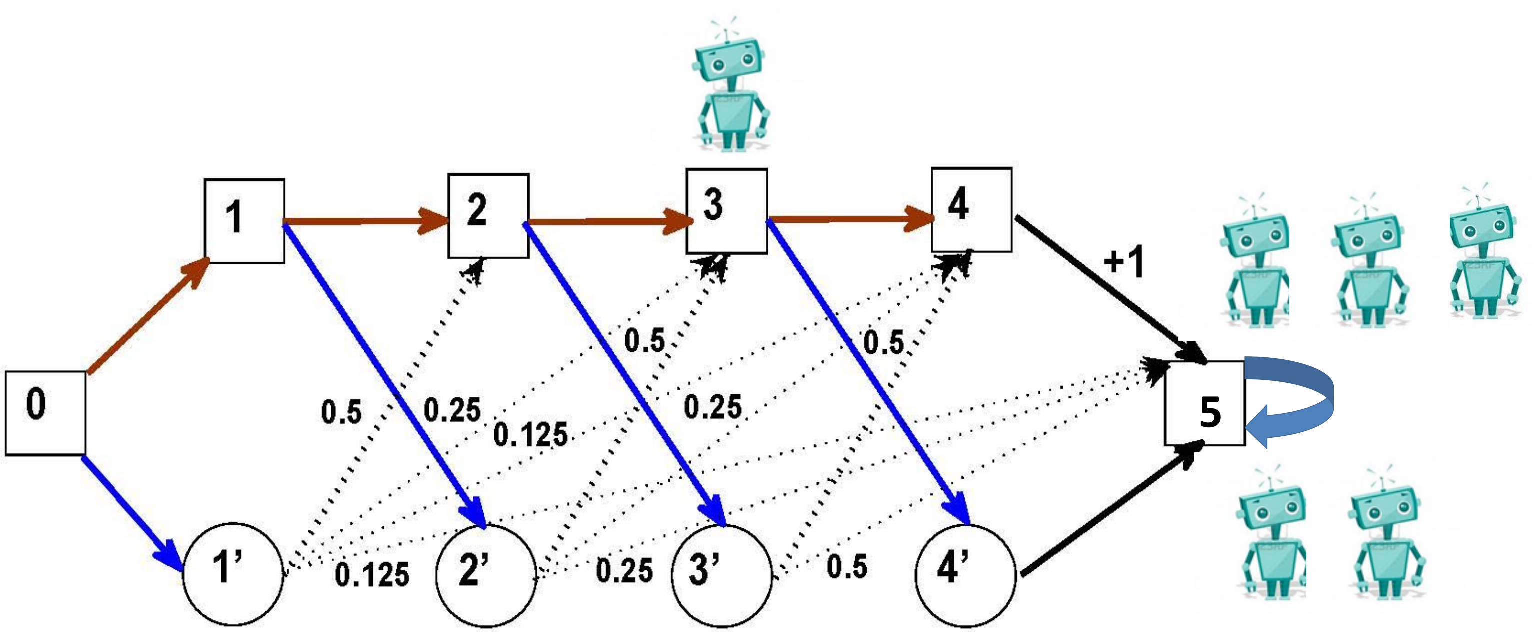
0.125
0.25
0.5

Time 4



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1'
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4'
+1
0.5
0.25
0.125
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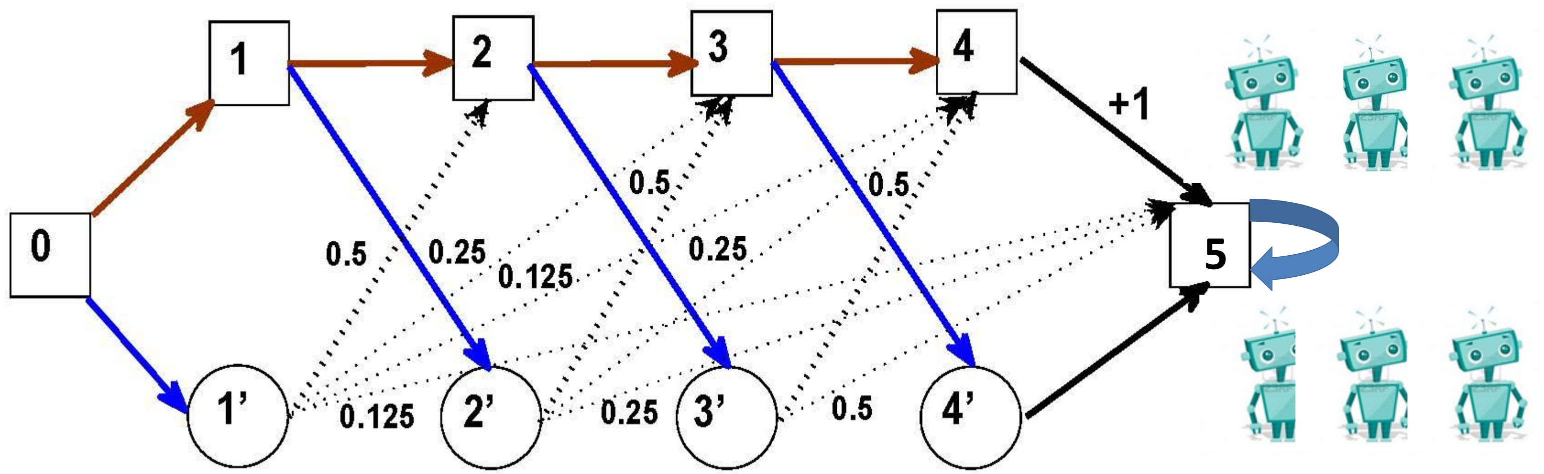
0.25
0.5
0.125
0.25
0.5

Time 5



Recall the optimal dual solution values are:

$$x_{0r}^* = 1,\ x_{1r}^* = 1 + \gamma,\ x_{2r}^* = 1 + \gamma + \gamma^2,\ x_{3b}^* = 1 + \gamma + \gamma^2 + \gamma^3,\ x_4^* = 1,$$

$$x_5^* = \frac{1 + 2\gamma + \gamma^2 + \gamma^3 + \gamma^4}{1 - \gamma}.$$

The Primal and Dual Problem of Optimization

- Every optimization problem is associated with another optimization problem called dual (the original problem is called primal).
- Every variable of the dual is the Lagrange multiplier associated with a constraint in the primal.
- The dual is max (min) if the primal is min (max)
- If the primal is a convex optimization problem, then the dual is also a convex optimization problem. Moreover, the two optimal objective values are equal (under mild technical assumptions).
- The optimal solution of the dual is the optimal Lagrange multiplier or shadow price vector of the primal.
- The above statements are also true if the constraint are nonlinear.

The Economic Interpretation of the Production Dual

Primal

$$\begin{array}{lllll} \max & x_1 + 2x_2 & & & \\ \text{s.t.} & x_1 & & & \le 1 \\ & & & x_2 & \le 1 \\ & x_1 & + & x_2 & \le 1.5 \\ & x_1, & & x_2 & \ge 0 \end{array}$$

Dual

$$\begin{array}{llllll} \min & y_1 + y_2 + 1.5y_3 & & & & \\ \text{s.t.} & y_1 & & + & y_3 & \ge 1 \\ & & y_2 & + & y_3 & \ge 2 \\ & y_1, & y_2, & & y_3 & \ge 0 \end{array}$$

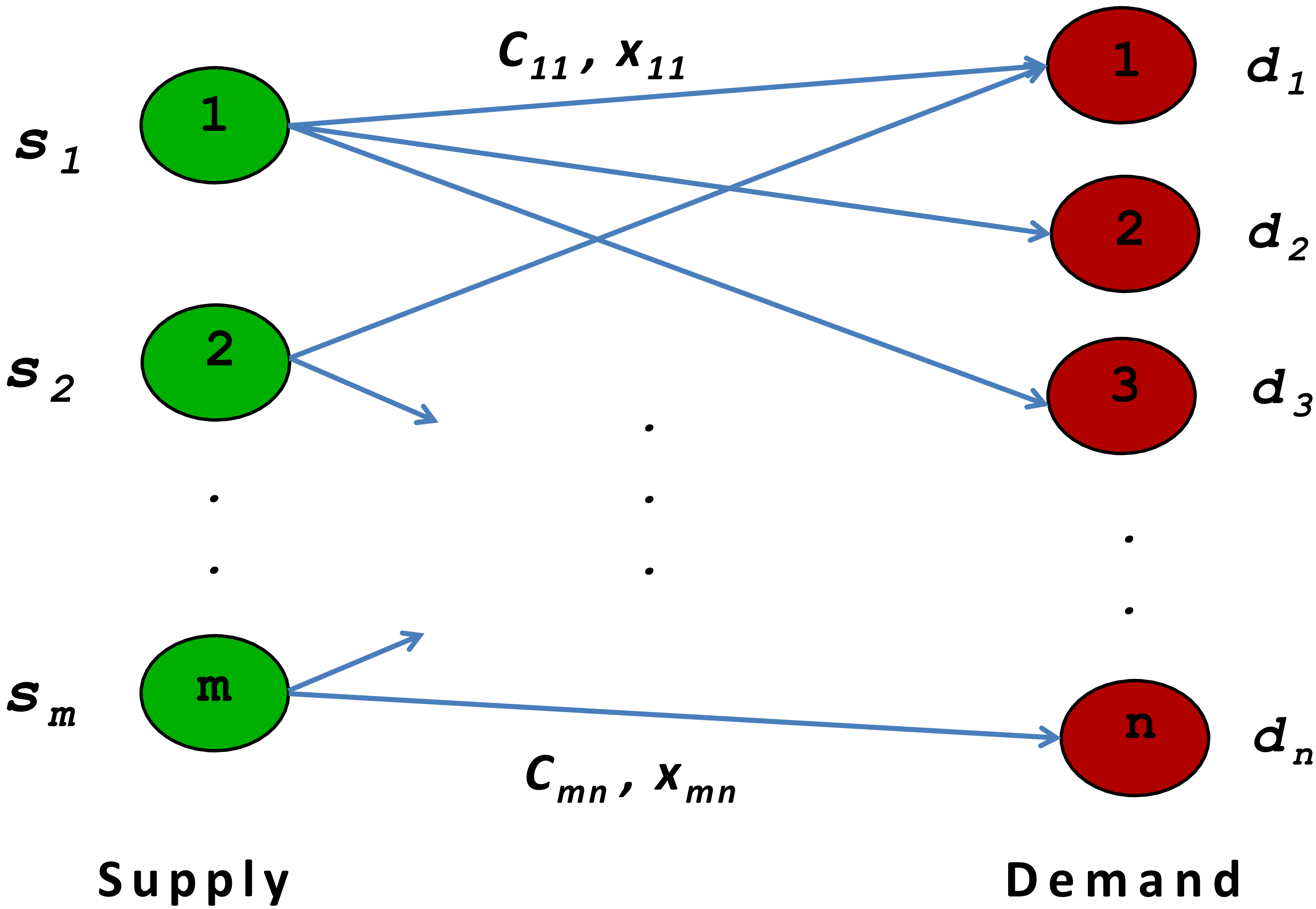
$\max\ \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$

$\min\ \mathbf{b}^T\mathbf{y}$ s.t. $A^T\mathbf{y} \ge \mathbf{c},\ \mathbf{y} \ge \mathbf{0}$

Acquisition Pricing:

- $\boldsymbol{y}$: prices of the resources
- $A^T\boldsymbol{y} \ge \boldsymbol{c}$: the prices are competitive for each product
- $\min\ \boldsymbol{b}^T\boldsymbol{y}$: minimize the total liquidation cost

The Transportation Dual



The Transportation Example

	1	2	3	4	Supply
1	12	13	4	6	500 u_1
2	6	4	10	11	700 u_2
3	10	9	12	4	800 u_3
Demand	400 v_1	900 v_2	200 v_3	500 v_4	20000

The Transportation Dual Interpretation

Primal

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = s_i, \quad \forall\, i = 1,\ldots,m \\
& \sum_{i=1}^{m} x_{ij} = d_j, \quad \forall\, j = 1,\ldots,n \\
& x_{ij} \ge 0, \quad \forall\, i, j
\end{aligned}
$$

Dual

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\
\text{s.t.} \quad & u_i + v_j \le c_{ij}, \quad \forall\, i, j
\end{aligned}
$$

Shipping Company's new charge scheme:

u_i: supply site unit charge

v_i: demand site unit charge

$u_i + v_j \le c_{ij}$: competitiveness

LP Duality Theorem

Theorem 1 *(Weak duality theorem) Let both primal feasible region F_p and dual feasible region F_d be non-empty. Then,*

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y} \qquad \textit{for all} \qquad \mathbf{x} \in F_p,\ \mathbf{y} \in F_d.$$

Proof: $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x} - (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T(\mathbf{c} - A^T\mathbf{y}) = \mathbf{x}^T\mathbf{r} \geq 0.$

This theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}$ the duality gap.

If the duality gap is zero, then $\mathbf{x}$ and $\mathbf{y}$ are optimal for the primal and dual, respectively! Is the reverse true?

LP Duality Theorem continued

Proved by the Simplex Method

Theorem 2 *(Strong duality theorem) Let both primal feasible region F_p and dual feasible region F_d be non-empty. Then, $\mathbf{x}^* \in F_p$ is optimal for (LP) and $\mathbf{y}^* \in F_d$ is optimal for (LD) if and only if the duality gap $\mathbf{c}^T\mathbf{x}^* - \mathbf{b}^T\mathbf{y}^* = 0$ (no need for technical assumptions).*

Corollary *If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.*

If one of (LP) or (LD) has no feasible solution, then the other is either unbounded or has no feasible solution. If one of (LP) or (LD) is unbounded then the other has no feasible solution.

Possible Combination of Primal and Dual

Dual \ Primal	F-B	F-UB	IF
F-B	☺		☹
F-UB			☹
IF	☹	☹	☹

Only in Nonlinear Optimization

$$
\begin{array}{llll}
\min & -x_1 & - & x_2 \\
\text{s.t.} & x_1 & - & x_2 = 1 \\
 & -x_1 & + & x_2 = 1 \\
 & x_1, & & x_2 \geq 0
\end{array}
$$

$$
\begin{array}{llll}
\max & y_1 & + & y_2 \\
\text{s.t.} & y_1 & - & y_2 \leq -1 \\
 & -y_1 & + & y_2 \leq -1
\end{array}
$$

Application of the Theorem: Alternative Systems

Consider the primal feasible system: $\{\mathbf{x} : A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. If it is infeasible, then the dual must be unbounded, that is, there exists a $\mathbf{y}$ in system

$$\{\mathbf{y} : A^T\mathbf{y} \leq \mathbf{0},\ \mathbf{b}^T\mathbf{y} > 0\}.$$

The reverse is also true. These two systems are an alternative pair: one and only one of the two is feasible.

Consider the dual feasible system: $\{\mathbf{y} : A^T\mathbf{y} \leq \mathbf{c}\}$. If it is infeasible, then the primal must be unbounded, that is, there exists an $\mathbf{x}$ in system

$$\{\mathbf{x} : A\mathbf{x} = \mathbf{0},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{c}^T\mathbf{x} < 0\}.$$

The reverse is also true. These two systems are also an alternative pair: one and only one of the two is feasible.

Recall the LP Optimality Condition

Check if a pair of primal ***x*** and dual ***y,*** with slack ***r***, is optimal:

$$\left\{ (\mathbf{x}, \mathbf{y}, \mathbf{r}) \in (R^n_+, R^m, R^n_+) : \begin{array}{lcl} \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} & = & \mathbf{0} \\ A\mathbf{x} & = & \mathbf{b} \\ A^T\mathbf{y} + \mathbf{r} & = & \mathbf{c} \end{array} \right\},$$

which is a system of linear inequalities and equations. Thus it is easy to verify whether or not a pair **(x, y, r)** is optimal by a computer.

These conditions can be classified as

- Primal Feasibility,
- Dual Feasibility, and
- Zero Duality Gap.