Hidden Linear Programming

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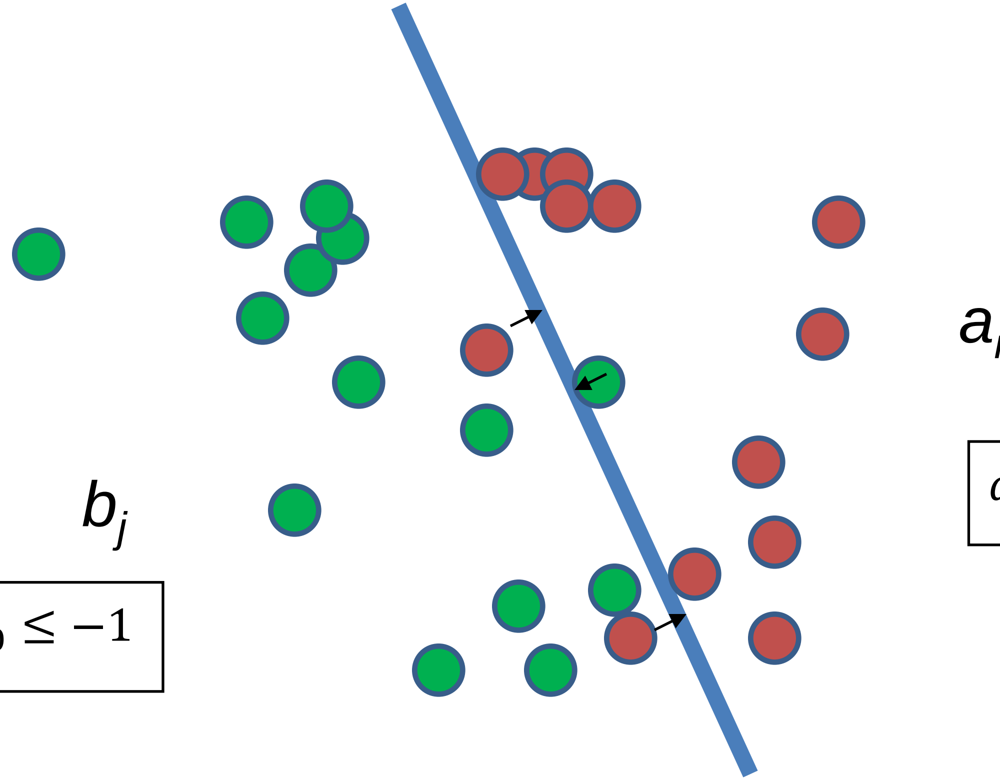
Read Chapter 1.1, 1.2, 2.1, 2.2, Appendices A, B&D in Textbook (hard copies would be available in the Book Store)

Today's Agenda

- Hidden LPs
  - Supporting Vector Machine when strict separation may not be possible
  - Air traffic landing time control
  - Financial Big-Data analysis
  - Combinatorial auction for information market
  - Reinforcement Learning/Markov Decision Process

Supporting Vector Machine Revisited

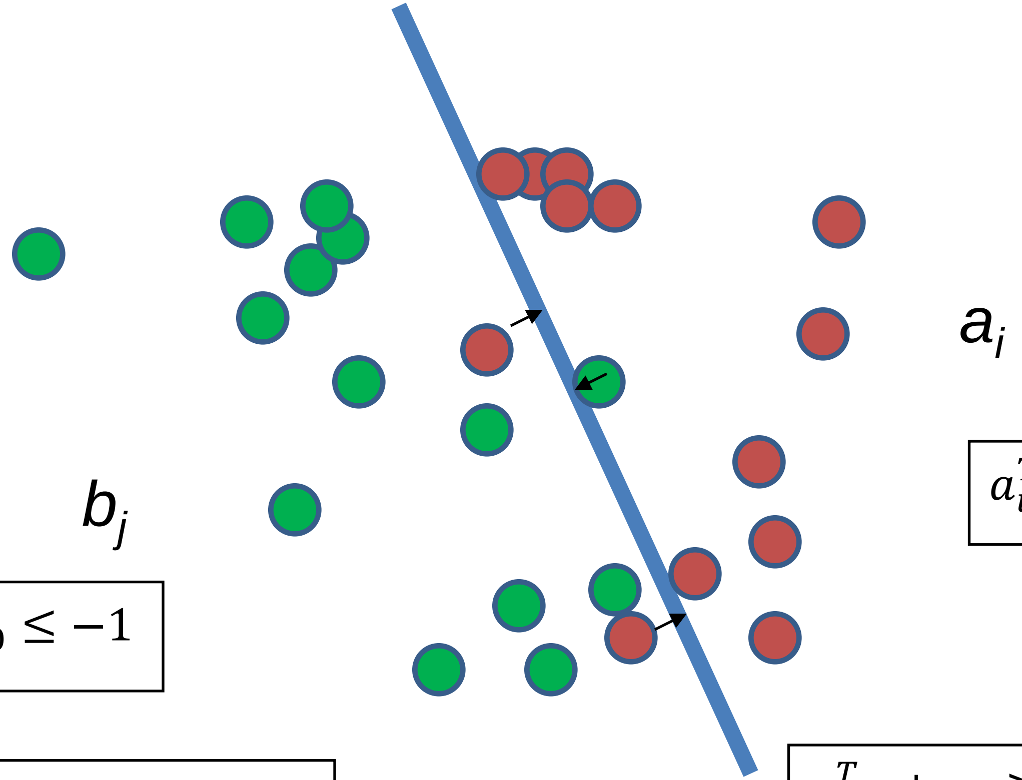


$$a_i^T x + x_0 \geq 1$$

$$b_j^T x + x_0 \leq -1$$

$$\text{minimize} \left\{ \textstyle\sum_i \max\left(1 - a_i^T x - x_0\,, 0\right) + \sum_j \max(b_j^T x + x_0 + 1, 0) \right\}$$

Supporting Vector Machine Revisited



$$a_i^T x + x_0 \geq 1$$

$$a_i^T x + x_0 \geq 1 - \delta'_i, \delta'_i \geq 0$$

$$b_j^T x + x_0 \leq -1$$

$$b_j^T x + x_0 \leq -1 + \delta''_j, \delta''_j \geq 0$$

$$\text{minimize} \quad \sum_i \delta'_i + \sum_j \delta''_j$$

How to Linearize the Max Function

Introduce an auxiliary variable *w*

$$\boxed{\max_{j=1,\ldots,m}\{\sum_i a_{ij}x_i\} = w}$$

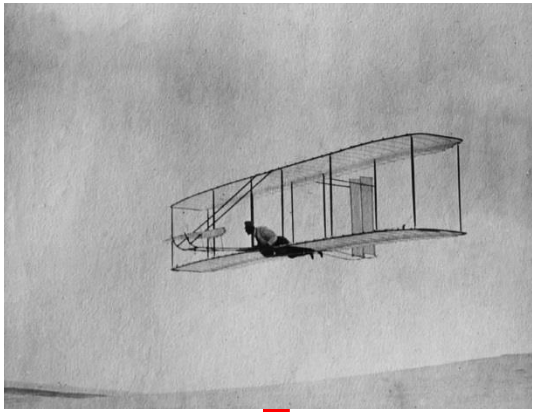
Relax it to linear inequalities

$$\boxed{\sum_i a_{ij}x_i \le w, j = 1, \ldots, m}$$

If *w* is minimized, the equality must hold

Air Traffic Control

PBS

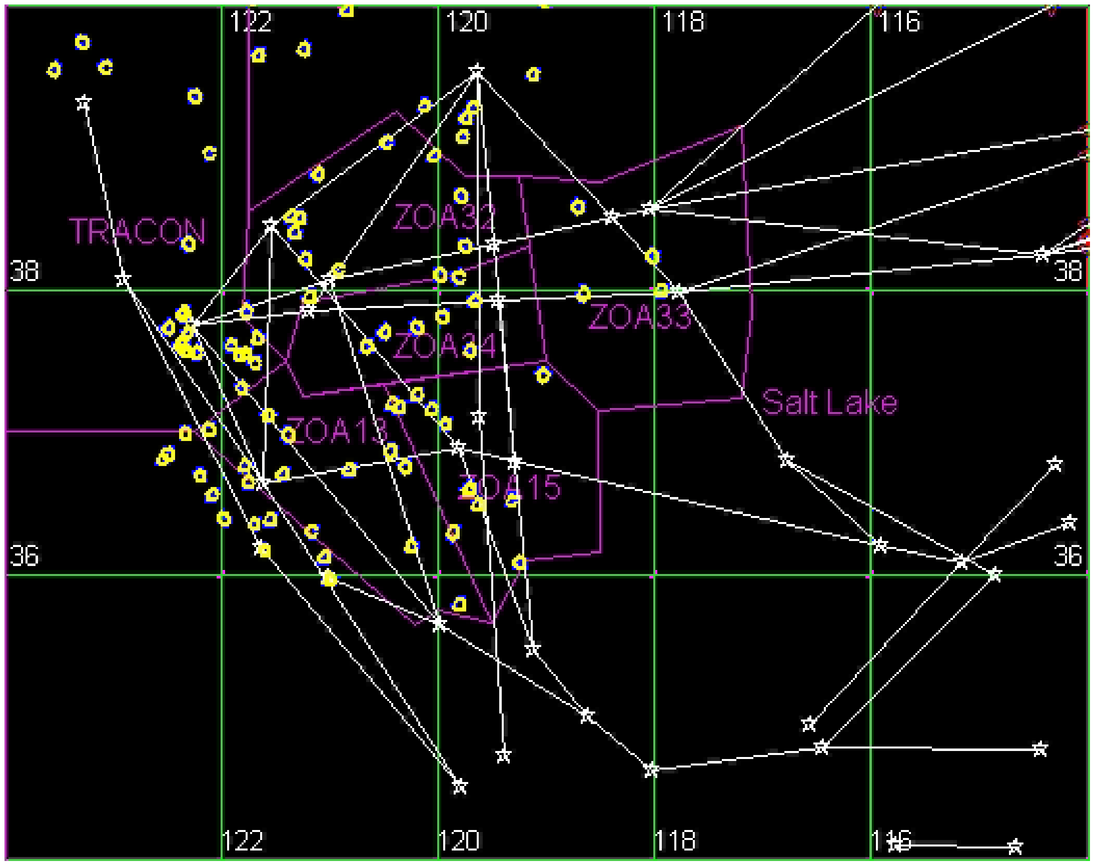


Nolan, Fundamentals of Air Traffic Control

Boeing

CNN

Oakland Center



Real data
(playback mode)

ETMS data
courtesy of NASA
Ames

Air Traffic Landing Control

- Air flight j, $j = 1, \ldots, n$, must arrive at the airport within the time interval $[a_j, b_j]$ in the order of 1, 2, ..., n.
- The airport wants to find the actual arrival time for each air plane such that the narrowest metering time (inter-arrival time between two consecutive airplanes) is the greatest.
- Let: t_j be the arrival time of flight j. Then

$$\begin{array}{ll} \text{maximize} & [\ \min_{j=1,\ldots,n-1}\{\, t_{j+1} - t_j \}\ ] \\ \text{s.t.} & a_j \le t_j \le b_j, j = 1, \ldots, n. \end{array}$$

This is not an LP problem!

How to Linearize the Min Function

Introduce an auxiliary variable Δ

$$\min_{j=1,\ldots,n-1}\{\, t_{j+1} - t_j \} = \Delta$$

Relax it to linear inequalities

$$t_{j+1} - t_j \geq \Delta, \qquad j = 1, \ldots, n-1.$$

If Δ is maximized, the equality must hold

$$\begin{array}{ll} \max & \Delta \\ \text{s.t.} & a_j \leq t_j \leq b_j, j = 1, \ldots, n, \\ & t_{j+1} - t_j - \Delta \geq 0, j = 1, \ldots, n-1. \end{array}$$

This is an LP problem!

Big Data: Business or Personal?

Build a model that will predict a probability for each credit card transaction indicating whether the transaction is business or personal related.

- There is no training data where particular transactions are identified as being personal, we used personal remittances as the best proxy
- On the transaction side, we focused on the industry code of each transaction as a key initial differentiator between transactions
- Developed a LP model to establish probabilities for each industry code that indicate the likelihood that dollars spent in that code will be personal spending.

Transaction Types by Industrial Codes

Industry Code	Description
995	CLUB - WAREHOUSE
25	DEPARTMENT STORE - MASS MERCHANDISER
728	GASOLINE/OIL COMPANY - NATIONAL DEALER
729	GASOLINE/OIL COMPANY - INDEPENDENT DEALER
429	SHOP - HOME IMPROVEMENT
415	DEPARTMENT STORE - FULL SERVICE
87	INTERNET TRAVEL
504	SHOP - ELECTRONIC GOODS
616	COMMUNICATION - CABLE & BROADCAST SERVICES
215	AUTO SERVICES - MOTOR RELATED SERVICES/DEALER
404	AUTO SERVICES - AUTO SALES & SERVICE
443	SHOP - SPORTING GOODS
457	SHOP - CHEMIST/PHARMACY
522	SHOP - FURNITURE
463	SHOP - JEWELRY
757	ENTERTAINMENT - TICKET AGENT - COMPANY
407	SHOP - CLOTHING - FAMILY
680	SHOP - COMPUTER HARDWARE
465	SHOP - LIQUOR STORE
400	AUTO SERVICES - VEHICLE ACCESSORIES
416	DEPARTMENT STORE - SPECIALITY
428	SHOP - HOME FURNISHINGS
414	SHOP - CLOTHING - WOMEN'S
793	TRAVEL - TOUR OPERATOR GENERAL
412	SHOP - CLOTHING - MEN'S & WOMEN'S
787	TRAVEL - NON - `AGENT RETAILER
447	SHOP - SHOES - MEN'S ONLY
427	SHOP - HARDWARE/DO IT YOURSELF
554	MAIL ORDER SELF IMPROVEMENT/BUSINESS SEMINARS
603	SERVICES - BEAUTY SHOPS/BEAUTICIAN

Business Analytics

For each of the industry codes, the model will determine a probability which indicates the likelihood that a transaction was personal.

Each Column represents an Industry Code

Personal Remittances

				…					
Account	**1**	**2**	**3**	…	**n**			**Actual**	
1	**\$156**	**\$0**	**\$87**		**\$25**			**\$200**	
2	**\$200**	**\$25**	**\$0**		**\$0**			**\$195**	
…	**\$0**	**\$134**	**\$35**		**\$60**			**\$210**	

Value of transactions in period

Model Example

For each of the industry codes, the model will determine a probability (in red) which indicates the likelihood that a transaction was personal. The goal is to minimize the sum of the squares of the differences (in blue).

Probability Personal

Each Column represents an Industry Code

Personal Remittances

	25%	10%	0%	…	5%				
Account	**1**	**2**	**3**	**…**	**n**		**Predicted**	**Actual**	**Difference**
1	**$156**	**$0**	**$87**		**$25**		**$244**	**$200**	**$44**
2	**$200**	**$25**	**$0**		**$0**		**$200**	**$195**	**$5**
…	**$0**	**$134**	**$35**		**$60**		**$230**	**$210**	**$20**

Value of transactions in period

LP Model?

Our model will determine the probability that a transaction from each industry code is personal in such a manner which will minimize the sum of the squared errors (between predicted personal remittances and actual personal remittances).

$$\text{Min} \quad \sum_i \; |\sum_j a_{ij}x_j - b_i|$$

$$\text{s.t.} \quad 0 \le x_j \le 1, \forall j.$$

- Let x_j be such a probability that a transaction is personal for industry code j
- $a_{i,j}$ – transaction amount for account i and industry code j
- b_i – amount paid by personal remit for account i
- $\sum_j a_{i,j} x_j$ – the expected personal expenses for account i
- We'd like to choose x_j such that $\sum_j a_{i,j} x_j$ matches b_i for ALL i

How to Linearize the Abs Function I

To dealing the abs function, we introduce auxiliary variables y_i

$$\boxed{|z_i| = y_i, i = 1, \ldots, m.}$$

Relax it to linear inequalities

$$\boxed{-y_i \leq z_i \leq y_i, i = 1, \ldots, m.}$$

If the sum of y_is is minimized, the equality must hold

$$\begin{array}{ll} \min & \sum_{i=1}^{m} y_i \\ \text{s.t.} & -y_i \leq \sum_j a_{ij} x_j - b_i \leq y_i, \forall i \\ & 0 \leq x_j \leq 1, \forall j. \end{array}$$

This is an LP problem!

$$\begin{array}{c} -y \leq Ax - b \leq y, \\ 0 \leq x \leq 1 \end{array}$$

How to Linearize the Abs Function II

Introduce auxiliary variables y'_i and y''_i

$$\boxed{z_i = y'_i - y''_i, y'_i \geq 0, y''_i \geq 0, i = 1, \dots, m.}$$

Relax it to linear inequalities

$$\boxed{\min |z_i| \quad \Leftrightarrow \quad \min y'_i + y''_i}$$

If the sum of y_is is minimized, the equality must hold

$$\begin{array}{ll} \min & \sum_{i=1}^{m} (y'_i + y''_i) \\ \text{s.t.} & Ax - b = y' - y'', \\ & 0 \leq x \leq 1, y' \geq 0, y'' \geq 0. \end{array}$$

This is an LP problem!

Mechanism for Information Market

- A place where information is aggregated via market for the primary purpose of forecasting events.

- **Why:**
  - Wisdom of the Crowds: Under the right conditions groups can be remarkably intelligent and possibly smarter than the smartest person.

    James Surowiecki
  - Efficient Market Hypothesis: financial markets are “informationally efficient”, prices reflect all known information
- **Market for Betting the World Cup Winner**
  - Assume 5 teams have a chance to win the World Cup: Argentina, Brazil, Italy, Germany and France

Optimizations for the Market

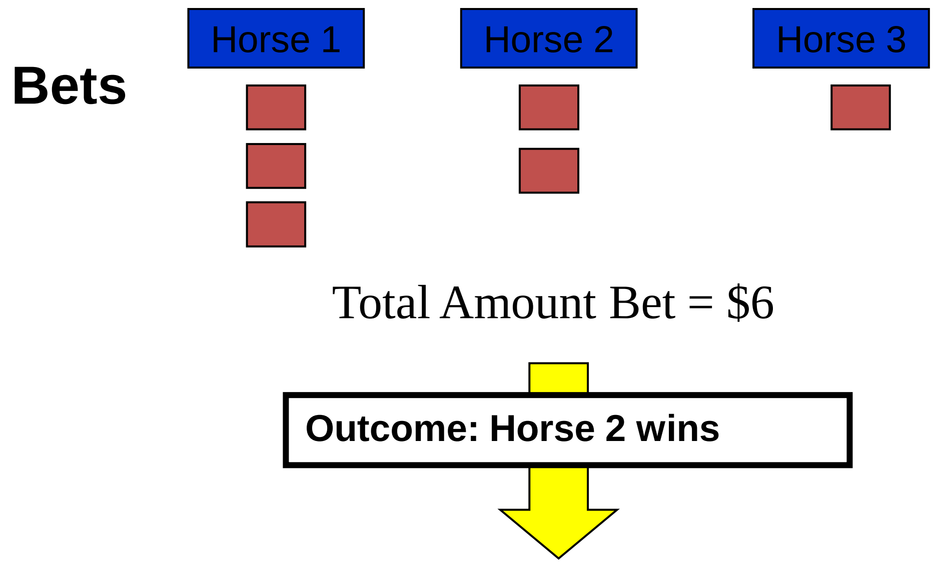
- **Double Auction:** Let participants trade directly with one another
  - Requires participants to find someone to take the other side of their order (i.e.: the complement of the set of teams which they have selected)
- **Centralized Market Maker**
  - Introduce a market maker who will accept or reject orders received from participants/traders
  - Market maker may be exposed to some risk
- **Problem:** How should the market maker fill orders in such a manner that he is not exposed to any financial risk?

Central Organization of the Market

- **Belief-based**
  - Central organizer will determine prices for each state based on his beliefs of their likelihood
  - This is similar to the manner in which fixed odds bookmakers operate in the betting world
  - Generally not self-funding
- **Pari-mutuel**
  - A self-funding technique popular in horseracing betting.

Pari-mutual Market Model 1

- Example: Pari-mutual Horseracing Betting



Winners earn $2 per bet plus stake back: Winners have stake returned then divide the winnings among themselves

More Abstract Market Model

- **Market for World Cup Winner**
  - We'd like to have a standard payout of $1 per share if a participant has a winning order.
- **List of Combinatorial Orders**

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

Market maker: Order fill - how many shares to sell for each order?

More Abstract Market Model

- Given m states that are mutually exclusive and exactly one of them will be realized at the maturity.
- An order is a bet on one or a combination of states
  - $(a_{i1}, a_{i2}, \ldots, a_{im})$: the entry value is 1 if the jth state is included in the winning basket and 0 other wise.
- with a price limit
  - π_i : the maximum price the participant is willing to pay for one share of the order
- and a share quantity limit
  - q_i : the maximum number of shares the participant is willing to buy.
- A contract agreement so that on maturity it is worth a notional one dollar per share if the order includes the winning state and worth 0 otherwise.

Pari-mutual Market Model 2

Order fill	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
x1	0.75	10	1	1	1		
x2	0.35	5				1	
x3	0.40	10	1		1		1
x4	0.95	10	1	1	1	1	
x5	0.75	5		1		1	

- Let x_i be the number of shares sell to order *i*.
- The revenue collected for the sale:

$$\boxed{\sum_i \pi_i x_i} \qquad \boxed{0.75x_1 + \ldots + 0.75x_5}$$

- The cost depends on which team wins:
  - If jth team wins (for example, if Brazil wins in the example):

$$\boxed{\sum_i a_{ij} x_i} \qquad \boxed{x_1 + x_4 + x_5}$$

- We consider the worse case cost and profit

$$\boxed{\max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\}} \Rightarrow \boxed{\max \ \left(\sum_i \pi_i x_i - \max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\}\right)}$$

LP Pari-mutual Market Mechanism

$$\begin{array}{lll} \max & \sum_i \pi_i x_i - \max_j \{ \sum_i a_{ij} x_i \} & \\ \text{s.t.} & 0 \le x_i \le q_i & \forall\ i = 1, \dots, n \end{array}$$

$$\begin{array}{lll} \max & \sum_i \pi_i x_i - w & \\ \text{s.t.} & \sum_i a_{ij} x_i \le w & \forall\ j \in S \\ & 0 \le x_i \le q_i & \forall\ i \in N \end{array}$$

Collected revenue

Worst-case cost

Cost if state j is realized

This is an LP problem; later you will learn that the optimal dual solution gives prices of each team

World Cup Betting Results

Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00

Compact Coefficients

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

π q A

Model in Matrix Form

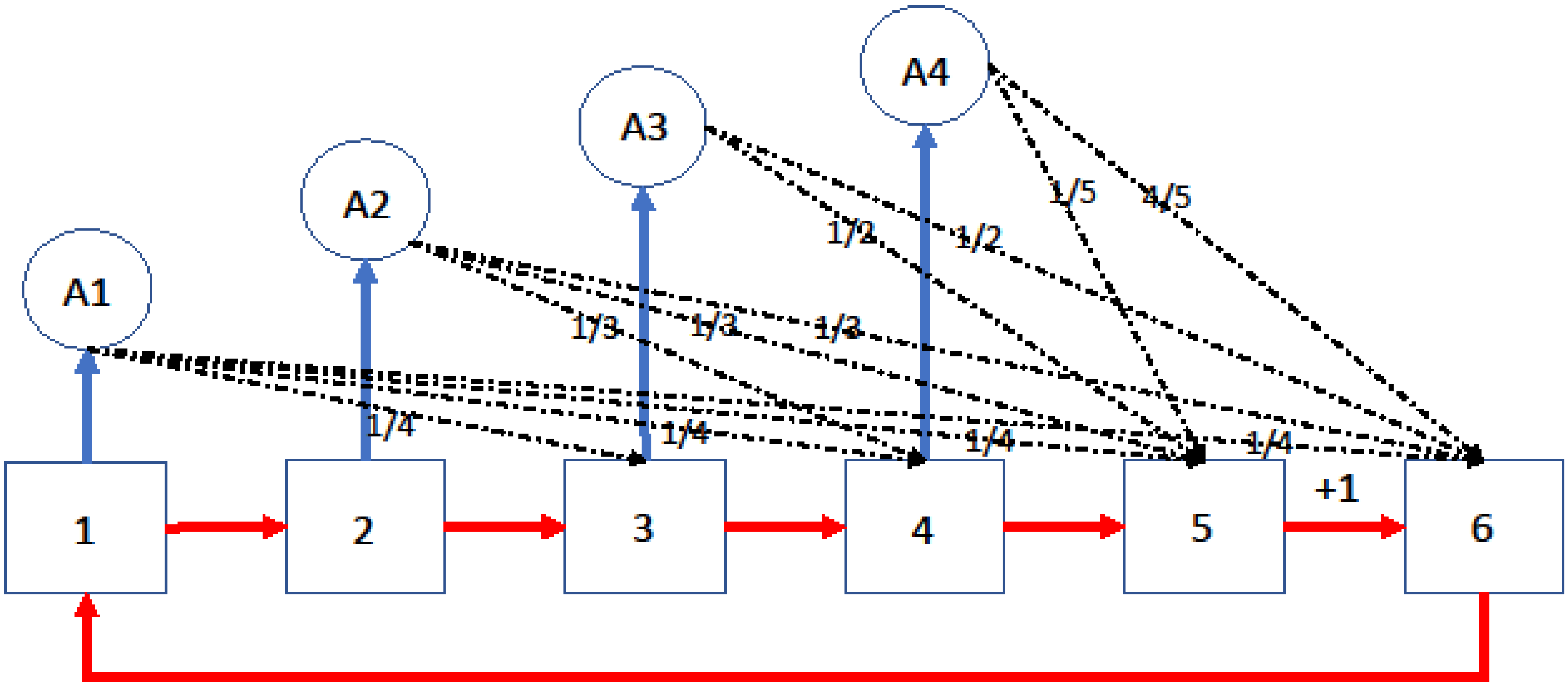
$$
\begin{array}{ll}
\max & \pi^T x \; - w \\
\text{s.t.} & A^T x - 1w \le 0, \\
& \quad x \qquad\; \le q, \\
& \quad x \qquad\; \ge 0
\end{array}
$$

1: vector of all ones

Reinforcement Learning and Markov Decision Process

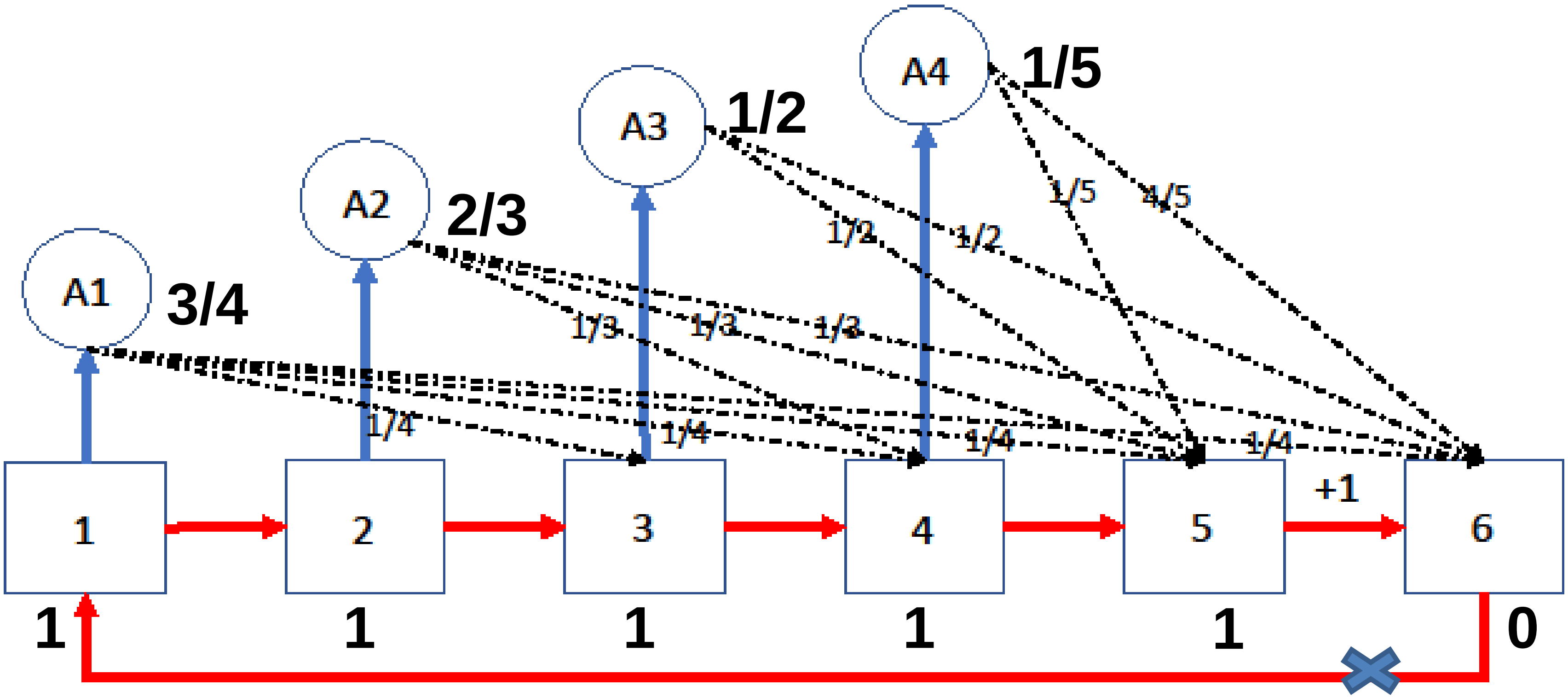
- Markov decision process provides a mathematical framework for modeling sequential decision-making in situations where outcomes are partly random and partly under the control of a decision maker, and it is called Reinforcement Learning lately.
- MDPs are useful for studying a wide range of optimization problems solved via dynamic programming, where it was known at least as early as the 1950s (cf. Shapley 1953, Bellman 1957).
- Modern applications include dynamic planning, social networking, and almost all other dynamic/sequential-decision-making problems in Mathematical, Physical, Management, Economics, and Social Sciences.
- MDP is characterized by States and Actions; and at each time step, the process is in a state and the decision maker chooses an action to optimize a long-term goal.

A Simple RL/MDP Problem: Maze Run



Each state *i* (in Square) is equipped with a set of actions A_i , and they are colored in red (status quo move), blue (shortcut move); and each of them incurs an immediate cost c_j. In this example, all actions have zero cost except the one from the state 4 (trap) to the final termination state 5 (Exit state which goes back to itself). Each action is associated with transition probability node (circle) with distribution vector P_j to all states.

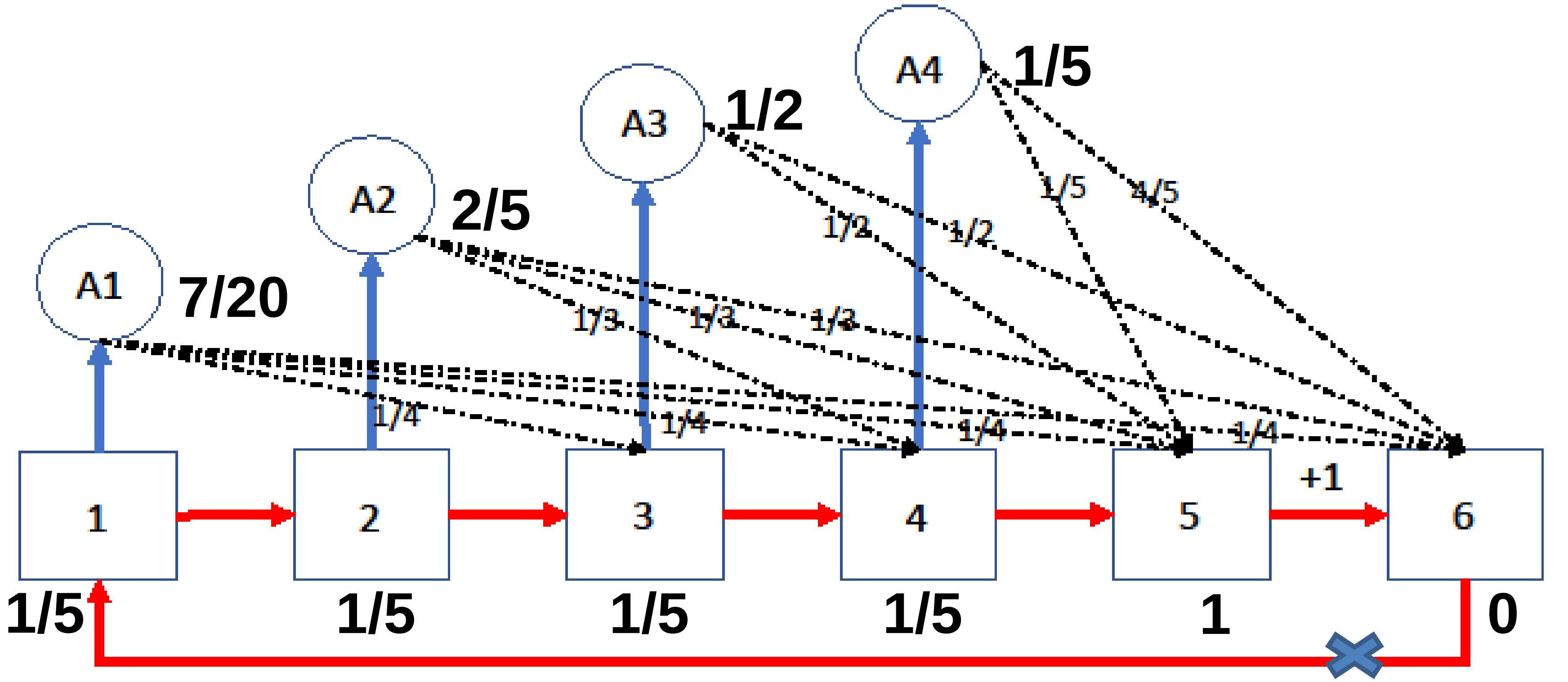
Cost-to-Go values of a Policy



A policy is a set of actions taken in each State at anytime, and it defines an expected Cost-to-Go value for each State (the overall present cost if starting from this very state). Assuming there is no discount and the current policy takes all-red actions, the corresponding expected cost-to-go state-values would be given above, together with expected values for blue-actions.

Clearly, this policy is not optimal…

Cost-to-Go values of another Policy

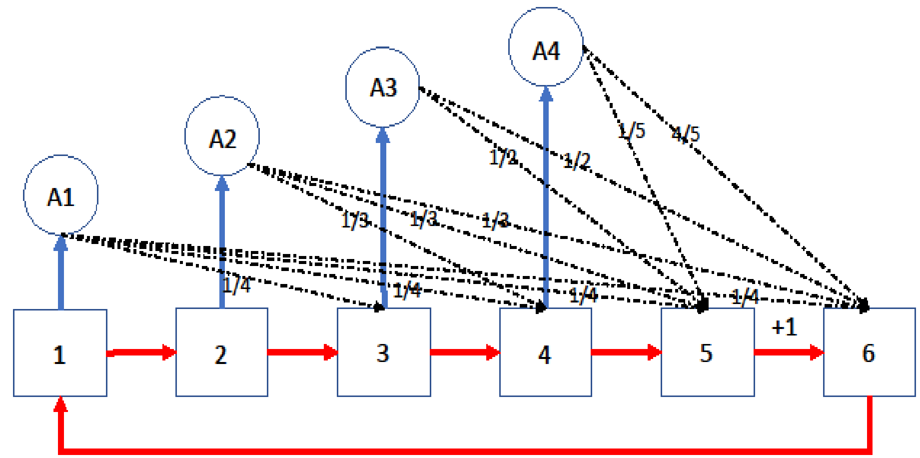


If the current policy is taking (red, red, red, blue, red) actions, the corresponding expected cost-to-go state-values would be given above, together with expected values for other actions. This policy is optimal.

An optimal policy is a policy that for each state there is no action-switch that results in a lower cost.

Cost-to-Go values of the Maze Run

- y_i: the expected overall present cost if stating from State i.
- State 5 is a trap
- State 6 is the exit state
- Each other state has two options:
  Go directly to the next state or
  a short-cut go to other states
  with uncertainties



- The cost-to-go values of the optimal policy with discount factor ϒ for this simple example should meet the following conditions

$$y_6 = 0 + \gamma y_1 , \quad y_5 = 1 + \gamma y_6$$

$$y_4 = \min\{0 + \gamma y_5 , 0 + \gamma(0.2 y_5 + 0.8 y_6\},$$

$$y_3 = \min\{0 + \gamma y_4 , 0 + \gamma(0.5 y_5 + 0.5 y_6)\}$$

$$y_2 = \min\{0 + \gamma y_3 , 0 + \gamma(0.33 y_4 + 0.33 y_5 + 0.33 y_6)\}$$

$$y_1 = \min\{0 + \gamma y_2, 0 + \gamma(0.25 y_3 + 0.25 y_4 + 0.25 y_5 + 0.25 y_6)\}$$

LP Formulation of the Maze Run

$$\max\ y_1 + y_2 + y_3 + y_4 + y_5 + y_6$$

s.t. $y_6 \le 0+ \gamma y_1$

$y_5 \le 1+ \gamma y_6$

$y_4 \le 0+ \gamma y_5$

$y_4 \le 0+ \gamma(0.2y_5+y_6)$

$y_3 \le 0+ \gamma y_4$

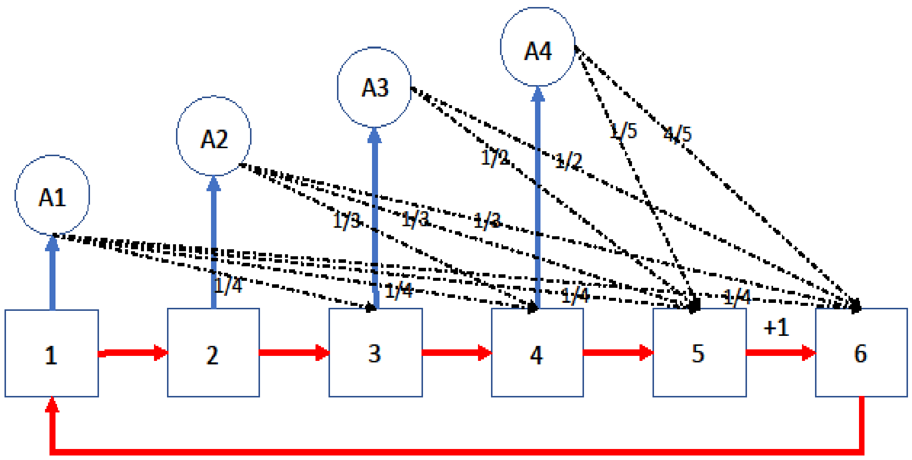
$y_3 \le 0+\gamma(0.5y_5+0.5y_6)$

$y_2 \le 0+ \gamma y_3$

$y_2 \le 0+ \gamma(0.33y_4+0.33y_5+0.33y_6)$

$y_1 \le 0+ \gamma y_2$

$y_1 \le 0+ \gamma(0.25y_3+0.25y_4+0.25y_5+0.25y_6)$



Cost-to-Go values and the LP formulation

- In general, let $y \in R^m$ represent the expected present cost-to-go values of the m states, respectively, for a given policy. Then, the cost-to-go vector of the optimal policy, with the discount factor γ, by Bellman's Principle is a Fixed Point:

$$y_i = \min\{ c_j + \gamma p_j^T y, j \in A_i \}, \forall i,$$

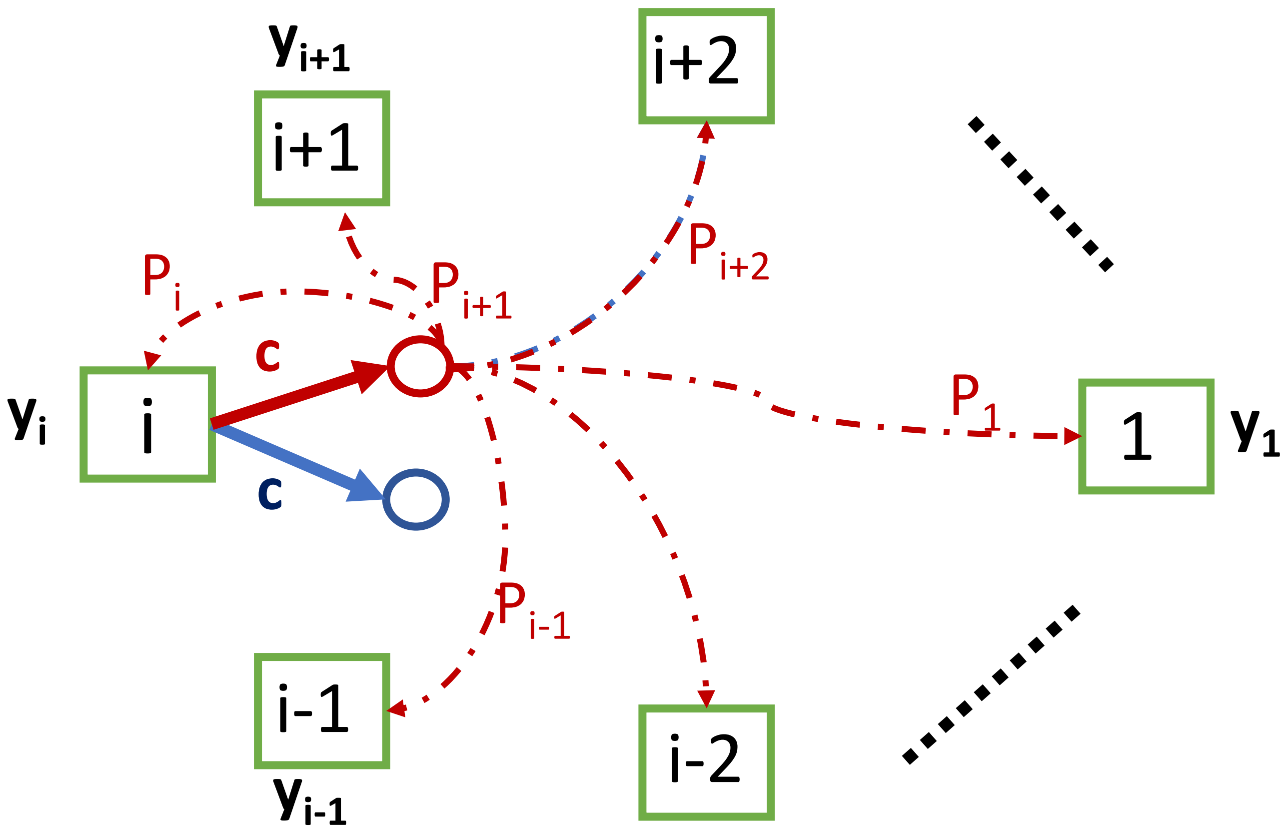
$$j_i = \arg\min\{ c_j + \gamma p_j^T y, j \in A_i \}, \forall i.$$

- Such a fixed-point computation can be formulated as an LP

$$\begin{array}{ll} \max & \sum_i y_i \\ \text{s.t.} & y_i \le c_j + \gamma p_j^T y, \forall j \in A_i; \forall i. \end{array}$$

- The maximization is trying to pushing up each yi to the highest possible so that it equal to min-argument. When the optimal y is found, one can then find the index of the original optimal action/policy using argmin.

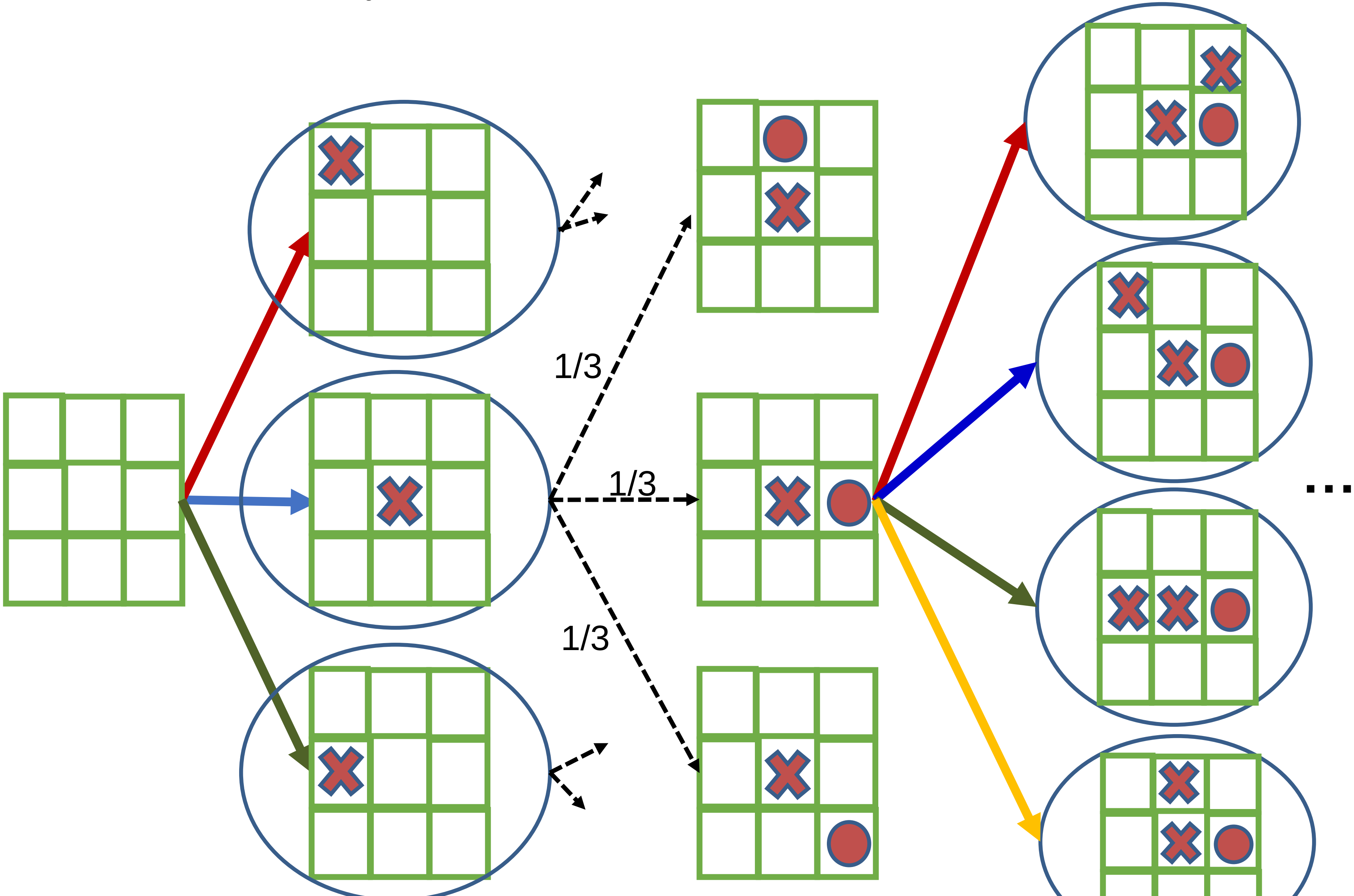
MDP/RL State/Action Environment



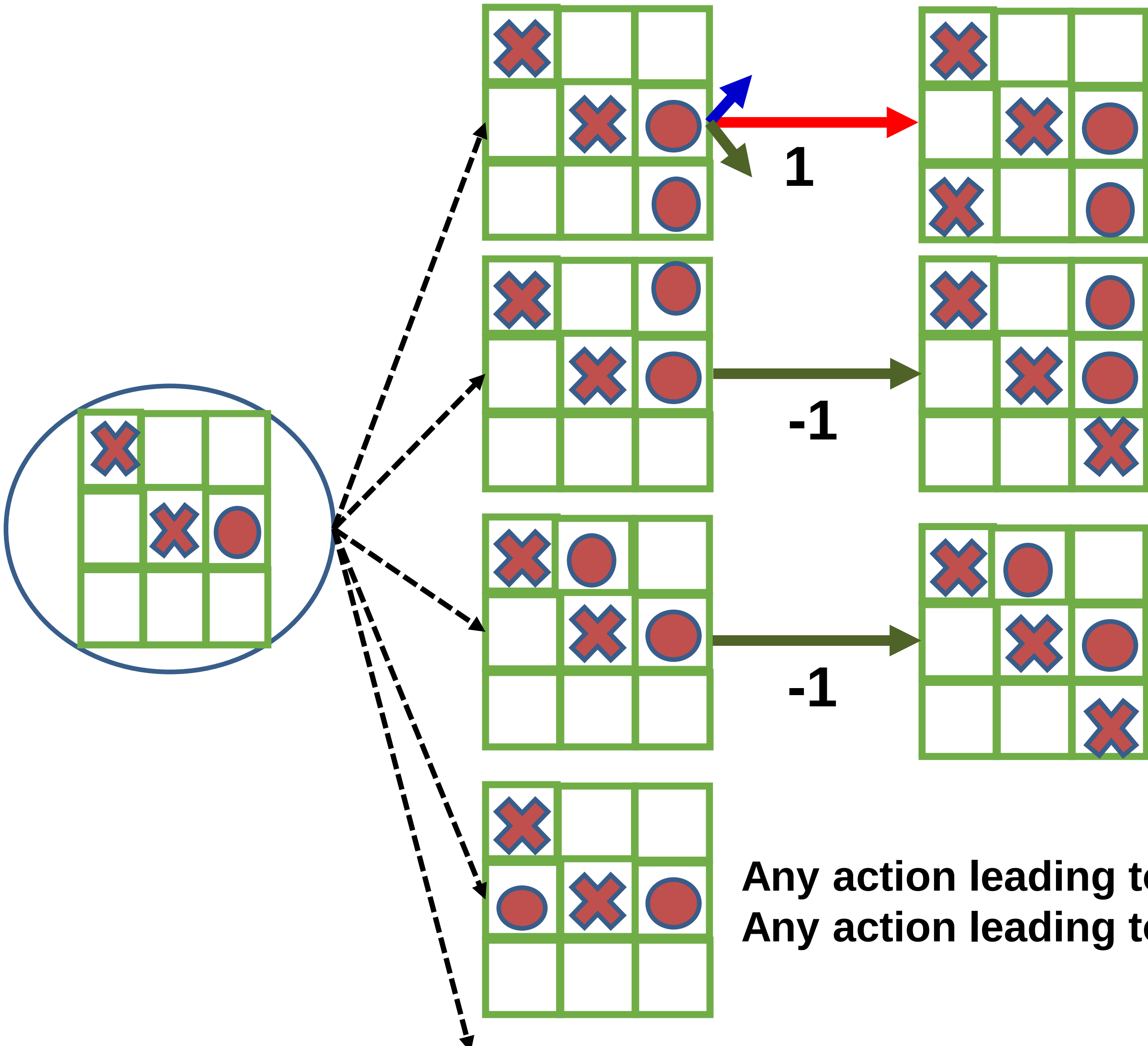
$$c \quad + \quad \gamma \quad p^T y$$

immediate cost expect future cost

States/Actions of Tic-Tac-Toe Game



Action Costs of Tic-Tac-Toe Game



Any action leading to win has cost -1
Any action leading to lose has cost 1