LP Optimality Conditions and Sensitivity Analyses

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Read Chapters 3.4

Recall the Simplex Method for Standard Equality Form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0 \end{aligned}$$

1. Start with a BFS basis *B* with **x**$_B$= $(A_B)^{-1}$ ***b***(≥**0**), **x**$_N$=**0**; compute **shadow (or dual) price** vector:

$$\mathbf{y}^T = \mathbf{c}^T_B (A_B)^{-1} \text{ or solve } \mathbf{y}^T A_B = \mathbf{c}^T_B$$

2. Calculate the **reduced cost** vector for non-basic variables

$$\boldsymbol{r}_N = \boldsymbol{c}^T_N - \boldsymbol{y}^T A_N \quad , \; (\, \boldsymbol{r}_B = \boldsymbol{0} \,)$$

If the reduced cost for every non-basic variable is nonnegative, then STOP: declare OPTIMAL

3. Dantzig Rule: select the most negative reduced cost variable, say x_e or column $(A_B)^{-1}\boldsymbol{A}_e$, as the entering variable (column), and using the minimum ratio to decide the outgoing variable (row). If the min-ratio is infinity, then STOP: declare UNBOUNDED

4.Update new basis (*B*) matrix inverse $(A_B)^{-1}$; or perform the pivot operations to update the tableau.

Go to Step 1

Summary of BFS Optimality Test/Condition

When a BFS with basis B, $\mathbf{x}_B$, is optimal?

$$\min \quad c^T x$$
$$\text{s.t.} \quad Ax = b,$$
$$x \geq 0$$

$$\mathbf{x}_B = (A_B)^{-1}\mathbf{b} \geq 0, \ \mathbf{x}_N = 0$$

$$\mathbf{r}^T = \mathbf{c}^T - \mathbf{y}^T A \ \geq \ 0$$

where the shadow-price/multiplier vector $\mathbf{y}^T = \mathbf{c}_B^T (A_B)^{-1}$.

Moreover $\quad$ OV = $\mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T (A_B)^{-1} \mathbf{b} = \mathbf{y}^T \mathbf{b}$

The existence of such a shadow-price/multiplier vector ***y*** is served as a certificate of the optimality of corner feasible solution ***x***. Such a **y** is also called optimal shadow-price vector.

Dos this optimal test/condition apply to any feasible solution ***x***?

The Optimality Condition

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0 \end{aligned}$$

Theorem A feasible solution ***x*** in the LP standard equality form is optimal if and only if there is an optimal shadow-price vector ***y*** such that:

$$\left\{ (\mathbf{x}, \mathbf{y}) \in (R^n, R^m) : \begin{array}{ll} \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{0} & \\ A\mathbf{x} & = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \\ A^T\mathbf{y} & \leq \mathbf{c} \end{array} \right\},$$

This is a system of linear inequalities and equations. Thus it is easy to verify whether or not a pair **(x, y)** is optimal by a computer.

An Equivalent Optimality Condition

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0 \end{aligned}$$

A feasible solution $\boldsymbol{x}$ in the LP standard equality form is optimal if and only if there are vectors $(\boldsymbol{y}, \boldsymbol{r})$ such that:

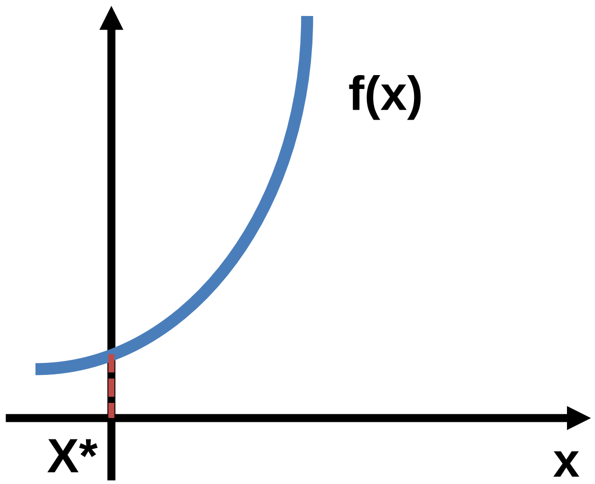
$$\left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{r}) \in (R^n, R^m, R^n) : \begin{array}{ll} \boldsymbol{r}^T \boldsymbol{x} = 0 & \\ A\mathbf{x} & = \mathbf{b}, \ \mathbf{x} \geq 0 \\ A^T \mathbf{y} + \mathbf{r} & = \mathbf{c}, \ \mathbf{r} \geq 0 \end{array} \right\},$$

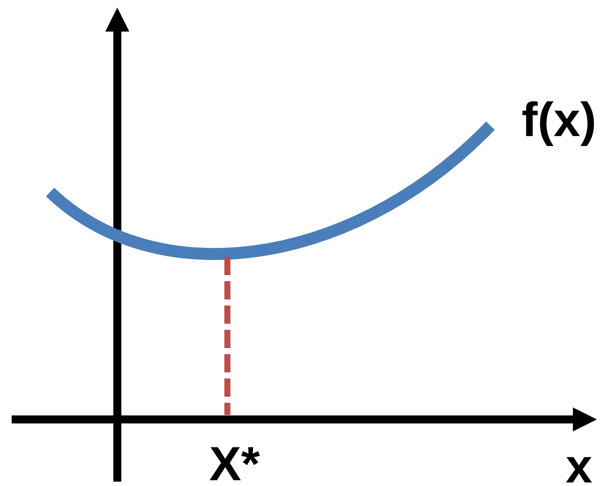
Since $\mathbf{r}^T\boldsymbol{x} = \sum_{j=1}^n r_j x_j$ and all entries are nonnegative, the condition implies that $r_j x_j = 0$ for all *j*; that is, for each *j*, at least one of r_j and x_j is 0. We often call this a complementarity property: two nonnegative vectors **r** and ***x*** are complementary to each other.

Physical Explanation of Complementarity Condition

Complementarity or Complementary-Slackness Phenomenon typically occurs when optimization with inequality constraints.

Consider min $f(x)$, s.t. $x \geq 0$





Two possible Scenarios:

$x^* = 0$ & $f'(0) \geq 0$ or $x^* > 0$ & $f'(x^*) = 0$

In both cases, the complementarity condition holds:
first, the derivative at the minimizer must be **nonnegative**;
second, it must be zero if the minimizer is in the interior of the constraint set, that is, the product of the derivative and the slack value mut be zero

What are in the Final Tableau

RC

- OV

	$c^T - y^T A$	$-y^T b$
	$(A_B)^{-1} A$	$(A_B)^{-1} b$

OS

Where price vector: $y^T = c_B^T (A_B)^{-1}$

Note that indexes in B can be in any order but still produce the same ***y***.

What are in the Final Tableau of the Production Problem

Recall that the initial tableau of production problem

$$\max \quad \boldsymbol{p}^T\boldsymbol{x}, \qquad \text{s.t.} \qquad R\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}$$

After transform it into the standard equality form:

$-\boldsymbol{p}^T$	0	0
R	I	$\boldsymbol{b}$

$-p^T-y^TR$	$-y^T$	$-y^Tb$
$(A_B)^{-1}R$	$(A_B)^{-1}$	$(A_B)^{-1}b$

RC → $-p^T-y^TR$

\- SP → $-y^T$

\- OV → $-y^Tb$

Basis Inverse → $(A_B)^{-1}$

OS → $(A_B)^{-1}b$

The 2-Product LP Problem Example

$$
\begin{array}{llllllll}
\min & -x_1 & -2x_2 & & & & \\
\text{s.t.} & x_1 & & +x_3 & & & = 1 \\
 & & x_2 & & +x_4 & & = 1 \\
 & x_1 & +x_2 & & & +x_5 & = 1.5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
$$

B	0	0	0	1	1	5/2
3	0	0	1	1	−1	1/2
2	0	1	0	1	0	1
1	1	0	0	−1	1	1/2

Final Tableau

A 4-Product Production Problem

$$
\begin{array}{llllllllll}
\max & 8x_1 & +14x_2 & +30x_3 & +50x_4 & & & & & \\
\text{subject to} & x_1 & +2x_2 & +10x_3 & +16x_4 & +x_5 & & & = & 800 \\
 & 1.5x_1 & +2x_2 & +4x_3 & +5x_4 & & +x_6 & & = & 1000 \\
 & 0.5x_1 & +0.6x_2 & +x_3 & +2x_4 & & & +x_7 & = & 340 \\
 & \mathbf{x} & \geq \mathbf{0}. & & & & & & &
\end{array}
$$

B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	-8	-14	-30	-50	0	0	0	0
5	1	2	10	16	1	0	0	800
6	1.5	2	4	5	0	1	0	1000
7	0.5	0.6	1	2	0	0	1	340

Final Tableau

B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	0	0	28	40	5	2	0	6000
2	0	1	11	19	1.5	-1	0	200
1	1	0	-12	-22	-2	2	0	400
7	0	0	-4	1.6	0.1	-0.4	1	20

r^T_N (row 0, columns x_3–x_6)

$-y^T$ (row 0, columns x_5–x_7)

$-OV$ (6000)

$(A_B)^{-1}A_N$ (rows 1–3, columns x_3–x_4)

$(A_B)^{-1}$ (rows 1–3, columns x_5–x_7)

x_B (right-hand column 200, 400, 20)

Interpretation of *y*: Shadow Price of RHS *b*

Given a BFS in the LP standard form with basis A_B

$$\mathbf{x}_B = (A_B)^{-1}\mathbf{b} > \mathbf{0}, \qquad \mathbf{x}_N = \mathbf{0},$$

so that small change in **b** does not change the optimal basis and the shadow price vector remains:

$$\mathbf{y}^T = c_B^T (A_B)^{-1}$$

At optimality, the OV is a function of **b:**

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T(A_B)^{-1}\mathbf{b} = \mathbf{y}^T\mathbf{b}.$$

Thus, when **b** is changed to **b**+Δ**b**, then the new OV

$$OV_+ = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T(A_B)^{-1}(\mathbf{b}+\Delta\mathbf{b}) = \mathbf{y}^T(\mathbf{b}+\Delta\mathbf{b}) = OV + \boxed{\mathbf{y}^T\Delta\mathbf{b}}$$

=Net Change

when the basis is unchanged.

$$OV(b) := \min \quad c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

*OV(**b**)* is a convex function of ***b***

and $\nabla OV(\mathbf{b}) = \mathbf{y}^*$

LP Shadow Price Properties

- The dimension of the shadow price (SP) vector equals the dimension of the right-hand-side (RHS) vector, or the number of linear constraints.
- In general, the optimal SP on a given active constraint is the rate of change in the optimal value (OV) of the objective as the RHS of the constraint increases in a interval, ceteris paribus.
- All inactive or nonbinding constraint have zero SP.
- In non-degenerate case, a small change in the RHS would change the OV and the optimal solution (OS), but not the basis and the optimal SPs.
- For infeasible problems, SPs represent which constraints need to be relaxed to make them feasible.

Interpretation of *r*: Reduced-Gradient Vector

Given a BFS in the LP standard form with basis A_B and its companion SP vector:

$$\mathbf{y}^T = \boldsymbol{c}_B^T(A_B)^{-1} \text{ and RC } \mathbf{r}_N^T=\mathbf{c}_N^T-\mathbf{y}^TA_N > 0$$

If $\mathbf{c}_N$ makes a small change, nothing would change. But if they reduced enough such that one of the reduced costs become negative, then the current BFS is no longer optimal.

On the other hand, if $\boldsymbol{c}_B$ makes a small change, say $\boldsymbol{c}_B$ is changed to $\boldsymbol{c}_B+\Delta\boldsymbol{c}_B$, then the new SP and OV

$$\mathbf{y}_+^T = (\boldsymbol{c}_B+\Delta\boldsymbol{c}_B)^T(A_B)^{-1} = \mathbf{y}^T+\Delta\mathbf{c}_B^T(A_B)^{-1}$$

$$OV_+=(\mathbf{y}^T+\Delta\mathbf{c}_B^T(A_B)^{-1})\mathbf{b}=OV+\Delta\mathbf{c}_B^T(A_B)^{-1}\mathbf{b}=OV+\boxed{\Delta\mathbf{c}_B^T x_B}$$

=Net Change

In general, the OV is a concave function of $\boldsymbol{c}$

LP Reduced Gradient Properties

- The dimension of the reduced-cost (RC) or reduced gradient vector equals the dimension of the objective coefficient vector or the number of decision variables.
- In general, the RC value of any non-basic variable is the amount the objective coefficient of that variable would have to change, ceteris paribus, in order for it to become a basic variable at optimality.
- All basic variables have zero RC.
- In non-degenerate case, a small change in the objective coefficients may change OV and optimal SP, but not the basis and OS.

B	0	0	28	40	5	2	0	6000
2	0	1	11	19	1.5	-1	0	200
1	1	0	-12	-22	-2	2	0	400
7	0	0	-4	1.6	0.1	-0.4	1	20

Post-Optimality Questions for the 4-Product Problem

By how much must the unit profit on variable 3 be increased before it would be profitable to manufacture it?

:This can be answered by simply checking the reduced cost of the final tableau for x_3, which is 28. The same question for x_4 is 40.

B	0	0	28	40	5	2	0	6000
2	0	1	11	19	1.5	-1	0	200
1	1	0	-12	-22	-2	2	0	400
7	0	0	-4	1.6	0.1	-0.4	1	20

Post-Optimality Questions for the 4-Product Problem

A competitor located next door has offered the manager additional Resource 1 at a rate of $4.50 per unit. Should he accept his offer?

: Easy, take it since the shadow price for this resource is $5☺

Suppose instead that the competitor offers the manager 250 units of Resource 1 for a total of $1,100, Should he accept his offer? (The manager can only accept or reject the extra 250 units.)

:We need to resolve the LP problem☹

B	0	0	28	40	5	2	0	6000
2	0	1	11	19	1.5	-1	0	200
1	1	0	-12	-22	-2	2	0	400
7	0	0	-4	1.6	0.1	-0.4	1	20

Post-Optimality Questions for the 4-Product Problem

The owner has approached the manager with a thought about producing a new type of product that would require 4 units of Resource 1, 4 units of Resource 2 and 1 unit of Resource 3. What should be the minimum unit profit of the new product such that it is to be manufactured?

: The answer is 28, why?

$$y = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}, \quad a_{new} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \quad y^T a_{new} = 28$$

The Lagrange Function and Theory

The (Penalty) **Lagrange Function or Lagrangian** was introduced for a constrained optimization problem to make it into a less constrained or unconstrained optimization problem.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax - b = 0, \\ & x \geq 0 \end{aligned}$$

For LP in the standard equality form, a partial Penalty Lagrangian is:

$$\min_x L(\boldsymbol{x},\boldsymbol{y}) = c^T\boldsymbol{x} - \mathbf{y}^T(A\boldsymbol{x}-\boldsymbol{b}), \text{ s.t. } \boldsymbol{x} \geq \boldsymbol{0};$$

where multipliers $\boldsymbol{y}$ in the Lagrange function play the role of penalty weights on equality constraint violations. One can adjust them whenever a constraint is violated at the minimizer of the Lagrange optimization problem (in the text book, we used **λ=-y**)

The goal is to select $\boldsymbol{y}$ such that the Lagrangian minimization yields an optimal solution $\boldsymbol{x}$ that is also optimal to the original constrained problem. It turns out such $\boldsymbol{y}$ must be an optimal shadow-price vector of the original LP problem.

$$
\begin{array}{llllllll}
\min & -x_1 & -2x_2 & & & & \\
\text{s.t.} & x_1 & & +x_3 & & & = 1 \\
 & & x_2 & & +x_4 & & = 1 \\
 & x_1 & +x_2 & & & +x_5 & = 1.5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
$$

For this example, the Lagrange function would be

$L(\boldsymbol{x},\boldsymbol{y})=-x_1-2x_2 - y_1(x_1+x_3-1) - y_2(x_2+x_4-1) - y_3(x_1+x_2+x_5-1.5)$

If we set $\boldsymbol{y}=(0,-1,-1)^T$ then $L(\boldsymbol{x}, (0,-1,-1)^T)= x_4 + x_5 -2.5.$

Therefore, minimize it subject to each variable to be nonnegative implies $x_4=0$ and $x_5=0$. Together with the equality constraints $A\boldsymbol{x}=\boldsymbol{b}$ in the original problem they yield the optimal solution for the original optimization problem.

Any other setting of $\boldsymbol{y}$ values in the Lagrangian will not make $\boldsymbol{x}$ possibly feasible to the original problem - either an x_j is negative or $A\boldsymbol{x}\neq\boldsymbol{b}$.

Optimality Condition via the Lagrangian Gradient

$$L(\mathbf{x},\mathbf{y}) = c^T\mathbf{x} - \mathbf{y}^T(A\mathbf{x}-\mathbf{b})$$

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax - b = 0, \\ & x \geq 0 \end{aligned}$$

Note the gradient vector with respect to $\mathbf{x}$ is

$$\nabla_x L(\mathbf{x},\mathbf{y}) = c - A^T\mathbf{y}$$

which we also called the reduced-cost vector $\mathbf{r}$. The function can be viewed as a "penalty" function for equality constraints.

Theorem A solution $\mathbf{x}$ in the LP standard equality form is optimal if and only if there is multiplier vector $\mathbf{y}$ such that the gradient vector with respect to $\mathbf{y}$ of the Lagrange function is a zero vector; and the gradient vector with respect to $\mathbf{x}$ is nonnegative and it is complementary to $\mathbf{x}$. The latter is because, consider $\min_x f(\mathbf{x})$ s.t. $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x}$ is minimal only if $\nabla f(\mathbf{x}) \geq \mathbf{0}$ and it is complementary to $\mathbf{x}$.

Whole Penalty Lagrangian

The Lagrange Function or Lagrangian can include variable nonnegativity as part of the constraints so that it becomes an unconstrained problem:

$$\min_x L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{r}) = c^T\boldsymbol{x} - \boldsymbol{y}^T(A\boldsymbol{x}-\boldsymbol{b}) - \boldsymbol{r}^T\boldsymbol{x}.$$

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax - b = 0, \\ & x \geq 0 \end{array}$$

where multipliers in ***y*** in the Lagrange function are penalty weights on equality constraint violations, and multipliers in ***r*** are penalty weights on inequality constraints $\boldsymbol{x \geq 0}$. Note that each penalty weight in ***r*** must be nonnegative since we only penalize the corresponding entry in ***x*** who becomes negative but no penalty otherwise.

Theorem A feasible solution ***x*** in the LP standard equality form is optimal if and only if there are multiplier vectors ***y*** and $\boldsymbol{r \geq 0}$ such that the gradient vector with respect to ***x*** of the Lagrange function is a zero vector and ***r*** is complementary to ***x***.