Linear Programming Duality and Dual Interpretations

Yinyu Ye

Stanford University and CUHKSZ (Sabbatical Leave)

Currently Visiting CUHK and HK PolyU

https://web.stanford.edu/class/msande211x/handout.shtml

Chapter 3.1-3.5

Recall of BFS Optimality Test/Condition

When a BFS with basis B, $\mathbf{x}_B$, is optimal?

$$\mathbf{x}_B = (A_B)^{-1}\mathbf{b} \geq 0,\ \mathbf{x}_N = 0$$

$$\mathbf{r}^T = \mathbf{c}^T - \mathbf{y}^T A \ \geq\ 0$$

where the shadow-price/multiplier vector $\boldsymbol{y}^T = \mathbf{c}_B{}^T (A_B)^{-1}$.

Moreover OV = $\mathbf{c}^T\boldsymbol{x} = \mathbf{c}_B{}^T\boldsymbol{x}_B = \mathbf{c}_B{}^T(A_B)^{-1}\boldsymbol{b} = \boldsymbol{y}^T\boldsymbol{b}$

The existence of such a shadow-price/multiplier vector $\boldsymbol{y}$ is served as a certificate of the optimality of corner feasible solution $\boldsymbol{x}$. Such a $\mathbf{y}$ is also called optimal shadow-price vector.

Dos this optimal test/condition apply to any feasible solution $\boldsymbol{x}$?

The Optimality Condition Theorem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0 \end{aligned}$$

Theorem A feasible solution ***x*** in the LP standard equality form is optimal if and only if there is an optimal shadow-price vector ***y*** such that:

$$\left\{ (\mathbf{x}, \mathbf{y}) \in (R^n, R^m) : \begin{array}{ll} \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = 0 & \\ A\mathbf{x} & = \mathbf{b}, \mathbf{x} \geq 0 \\ A^T\mathbf{y} & \leq \mathbf{c} \end{array} \right\},$$

This is a system of linear inequalities and equations. Thus it is easy to verify whether or not a pair **(x, y)** is optimal by a computer.

Sketch Proof of The Optimality Condition Theorem

Consider any vector y who satisfies

$$A^T y \le c.$$

Then for any feasible solution x in the LP standard equality form, we must have

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \ge 0 \end{aligned}$$

$$c^T x - b^T y = c^T x - (Ax)^T y = c^T x - y^T (Ax) = (c^T - y^T A) x \ge 0.$$

That is, the value $b^T y$ is a lower bound on any feasible objective value $c^T x$.

Thus, if $c^T x = b^T y$, $c^T x$ must be the minimal among all possible feasible solution x.
(Of course, $b^T y$ must be maximal among all possible y such that $A^T y \le c$, which is called the dual program; more on this later.)

An Equivalent Optimality Condition

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0 \end{array}$$

A feasible solution ***x*** in the LP standard equality form is optimal if and only if there are vectors (***y, r***) such that:

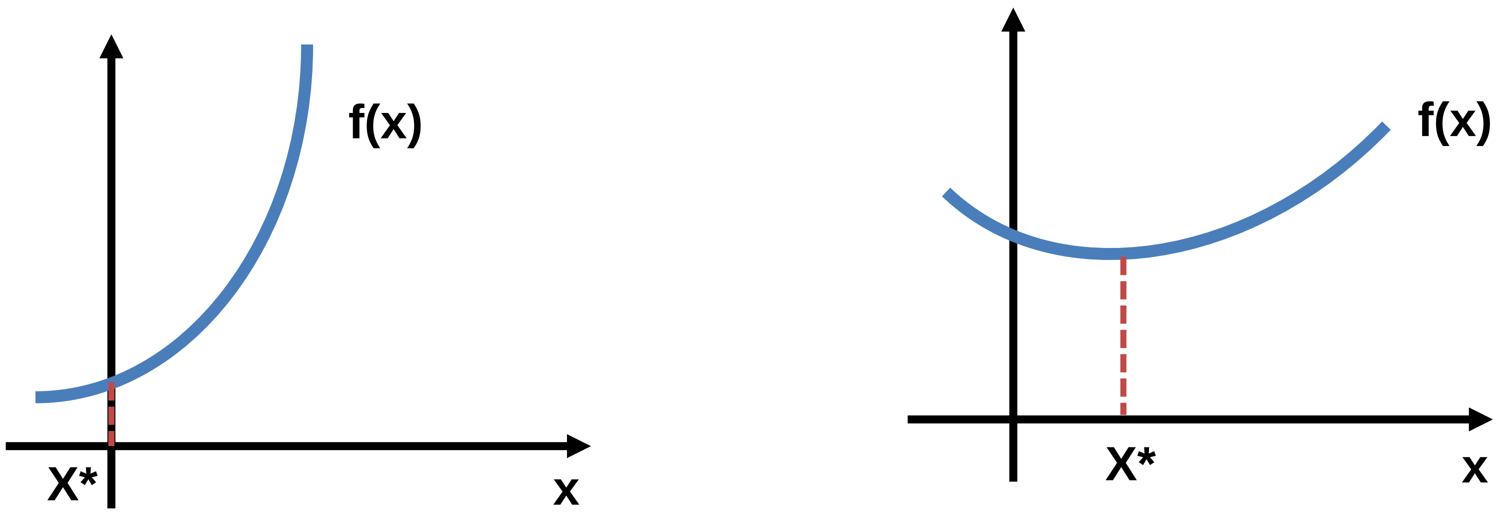
$$\left\{ (\mathbf{x}, \mathbf{y}, \mathbf{r}) \in (R^n, R^m, R^n): \begin{array}{ll} \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = 0 \text{ or } \mathbf{r}^T\mathbf{x} = \mathbf{0} & \\ A\mathbf{x} & = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0} \\ A^T\mathbf{y} + \mathbf{r} & = \mathbf{c}, \ \mathbf{r} \geq \mathbf{0} \end{array} \right\},$$

Since $\mathbf{r}^T\mathbf{x} = \sum_{j=1}^{n} r_j x_j$ and all entries are nonnegative, the condition implies that $r_j x_j = 0$ for all *j*; that is, for each *j*, at least one of r_j and x_j is 0. We often call this a complementarity property: two nonnegative vectors **r** and ***x*** are complementary to each other.

Physical Explanation of Complementarity Condition

Complementarity or Complementary-Slackness Phenomenon typically occurs when optimization with inequality constraints.

Consider min f(x), s.t. $x \geq 0$



Two possible Scenarios:

$x^*= 0$ & $f'(0) \geq 0$ or $x^*>0$ & $f'(x^*)=0$

In both cases, the complementarity condition holds:
first, the derivative at the minimizer must be **nonnegative**;
second, it must be zero if the minimizer is in the interior of the constraint set, that is, the product of the derivative and the slack value mut be zero

Complementary Slackness in World Cup Betting

Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

Shadow State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00

Interpretation of *y*: Shadow Price Vector of RHS *b*

Given a BFS in the LP standard form with basis A_B

$$\mathbf{x}_B = (A_B)^{-1}\mathbf{b} > \mathbf{0}, \qquad \mathbf{x}_N = \mathbf{0},$$

so that small change in **b** does not change the optimal basis and the shadow price vector remains:

$$\mathbf{y}^T = c_B^T(A_B)^{-1}$$

At optimality, the OV is a function of **b:**

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T(A_B)^{-1}\mathbf{b} = \mathbf{y}^T\mathbf{b}.$$

Thus, when **b** is changed to **b**+Δ**b**, then the new OV

$$OV_+ = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T(A_B)^{-1}(\mathbf{b}+\Delta\mathbf{b}) = \mathbf{y}^T(\mathbf{b}+\Delta\mathbf{b}) = OV + \boxed{\mathbf{y}^T\Delta\mathbf{b}}$$

=Net Change

when the basis is unchanged.

$$OV(b) := \min \quad c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

OV(***b***) is a convex function of ***b*** and $\nabla OV(\mathbf{b}) = \mathbf{y}^*$

Summary of Shadow Price (Lagrange Multiplier, Dual Variable)

- Each constraint is associated with a shadow price, also called Lagrange multiplier or dual variable
- They are used to certify whether or not a feasible solution is optimal.
- At an optimal solution, all inactive constraints have zero-valued Lagrange multiplier (called complementarity)
- At optimality, the Lagrange multiplier on a given active constraint is the rate of change in the optimal value (OV) as the RHS of the constraint increases with all other data held fixed.
- The reduced cost can be viewed as the Lagrange multiplier of the nonnegative constraint; a BFS is minimal if all reduced costs become nonnegative.

Recall in the LP production example, the BFS with $B = \{1, 2, 3\}$ is optimal with $x = (½, 1, ½, 0, 0)^T$ and $\mathbf{y} = (0, -1, -1)^T$

$$\begin{array}{llllllll}
\min & -x_1 & -2x_2 & & & & \\
\text{s.t.} & x_1 & & +x_3 & & & = 1 \\
& & x_2 & & +x_4 & & = 1 \\
& x_1 & +x_2 & & & +x_5 & = 1.5 \\
& x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}$$

The current OV= -2.5

- If b_1 is increased or decreased a little, does OV change?
- If b_2 is increased or decreased a little, does OV change? How much?
- If b_3 is increased or decreased a little, does OV change? How much?

This is called sensitivity analyses and an economical interpretation of $\mathbf{y}$.

The Primal and Dual Problem of Optimization

- Every optimization problem is associated with another optimization problem called dual (the original problem is called primal).
- Every variable of the dual is the Lagrange multiplier associated with a constraint in the primal.
- The dual is max (min) if the primal is min (max)
- If the primal is a convex optimization problem, then the dual is also a convex optimization problem. Moreover, the two optimal objective values are equal (under mild technical assumptions).
- The optimal solution of the dual is the optimal Lagrange multiplier or shadow price vector of the primal.
- The above statements are also true if the constraint are nonlinear.

Systematic Way to Construct the LP Dual

obj. coef. Vector	right-hand-side
right-hand-side	obj. coef. vector
A	A^T
Max model	Min model
$x_j \geq 0$	jth constraint $\geq$
$x_j \leq 0$	jth constraint $\leq$
x_j free	jth constraint $=$
ith constraint $\leq$	$y_i \geq 0$
ith constraint $\geq$	$y_i \leq 0$
ith constraint $=$	y_i free

The dual of the dual is the primal: either side can be the primal

The Economic Interpretation of the Production Dual

Primal

$$\begin{array}{lll} \max & x_1 + 2x_2 & \\ \text{s.t.} & x_1 & \le 1 \\ & \quad\quad x_2 & \le 1 \\ & x_1 + x_2 & \le 1.5 \\ & x_1, \quad x_2 & \ge 0 \end{array}$$

max $\mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$

Dual

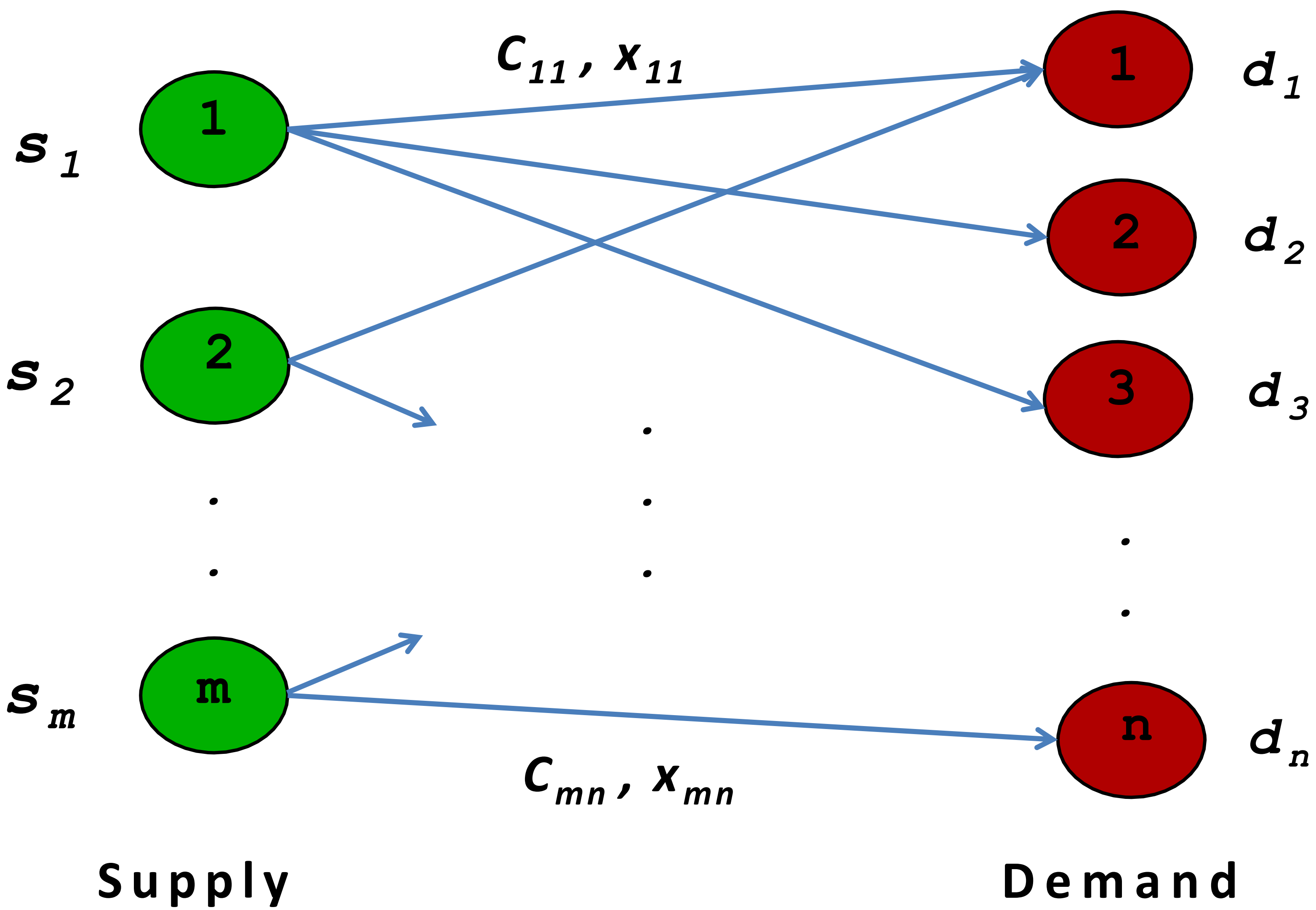
$$\begin{array}{lll} \min & y_1 + y_2 + 1.5y_3 & \\ \text{s.t.} & y_1 \quad\quad + \ y_3 & \ge 1 \\ & \quad\quad y_2 + \ y_3 & \ge 2 \\ & y_1, \ y_2, \ y_3 & \ge 0 \end{array}$$

min $\mathbf{b}^T\mathbf{y}$ s.t. $A^T\mathbf{y} \ge \mathbf{c},\ \mathbf{y} \ge \mathbf{0}$

Acquisition Pricing:

- $\boldsymbol{y}$: prices of the resources
- $A^T\boldsymbol{y} \ge \boldsymbol{c}$: the prices are competitive for each product
- min $\boldsymbol{b}^T\boldsymbol{y}$: minimize the total liquidation cost

The Transportation Dual



The Transportation Example

	1	2	3	4	Supply
1	12	13	4	6	500 u_1
2	6	4	10	11	700 u_2
3	10	9	12	4	800 u_3
Demand	400 v_1	900 v_2	200 v_3	500 v_4	20000

The Transportation Dual Interpretation

Primal

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = s_i, \quad \forall\, i = 1,\ldots,m \\ & \sum_{i=1}^{m} x_{ij} = d_j, \quad \forall\, j = 1,\ldots,n \\ & x_{ij} \geq 0, \quad \forall\, i, j \end{aligned}$$

Dual

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\ \text{s.t.} \quad & u_i + v_j \leq c_{ij}, \quad \forall\, i, j \end{aligned}$$

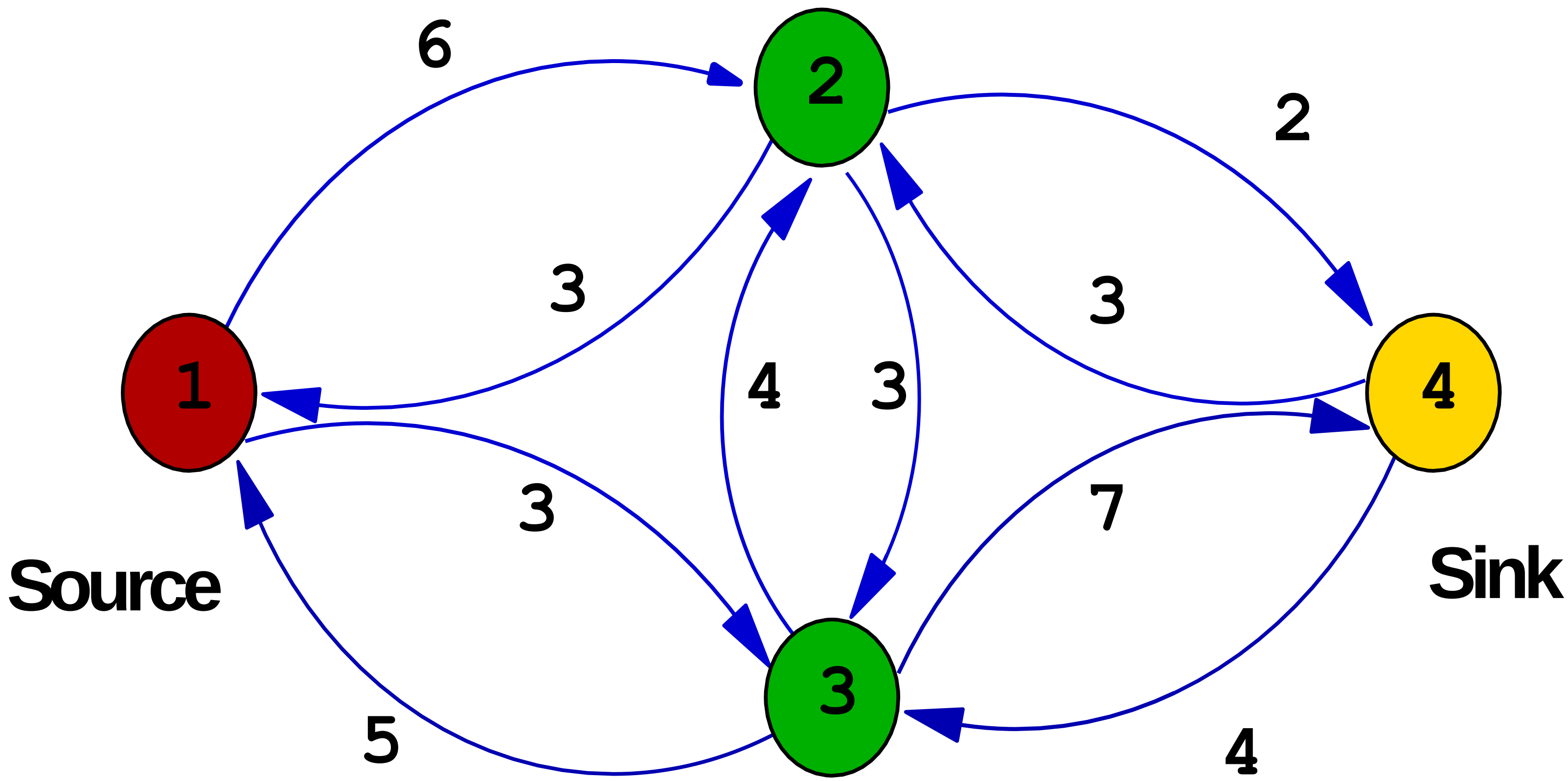
Shipping Company's new charge scheme:

u_i: supply site unit charge

v_i: demand site unit charge

$u_i + v_j \leq c_{ij}$: competitiveness

Look at a Max-Flow Problem



The Primal Formulation

Let x_{ij} be the flow rate from node i to node j. Then the problem can be formulated as

$$
\begin{array}{lll}
\max & x_{41} & \\
\text{s.t.} & x_{21} + x_{31} + x_{41} - x_{12} - x_{13} & = 0, \\
& x_{12} + x_{32} + x_{42} - x_{21} - x_{23} - x_{24}\, x_{13} & = 0, \\
& +\; x_{23} + x_{43} - x_{31} - x_{32} - x_{34}\, x_{24} + & = 0, \\
& x_{34} - x_{41} - x_{42} - x_{43} & = 0, \\
& x_{ij} & \le k_{ij}, \quad \forall (i, j) \in A, \\
& x_{ij} & \ge 0, \quad \forall (i, j) \in A.
\end{array}
$$

y_1

y_2

y_3

y_4

z_{ij}

Corresponding Dual variables

The Dual of Max-Flow: the Min-Cut Problem

Corresponding Primal variables

		Corresponding Primal variables
min	$\sum_{(i,j)\in A} k_{ij} z_{ij}$	
s.t.	$y_1 - y_4 = 1,$	x_{41}
	$-y_1 + y_2 + z_{12} \ge 0,$	x_{12}
	$-y_1 + y_3 + z_{13} \ge 0,$	x_{13}
	…	…
	$-y_2 + y_4 + z_{24} \ge 0,$	x_{24}
	$-y_3 + y_4 + z_{34} \ge 0,$	x_{34}
	$z_{ij} \ge 0, \ \forall (i, j) \in A.$	

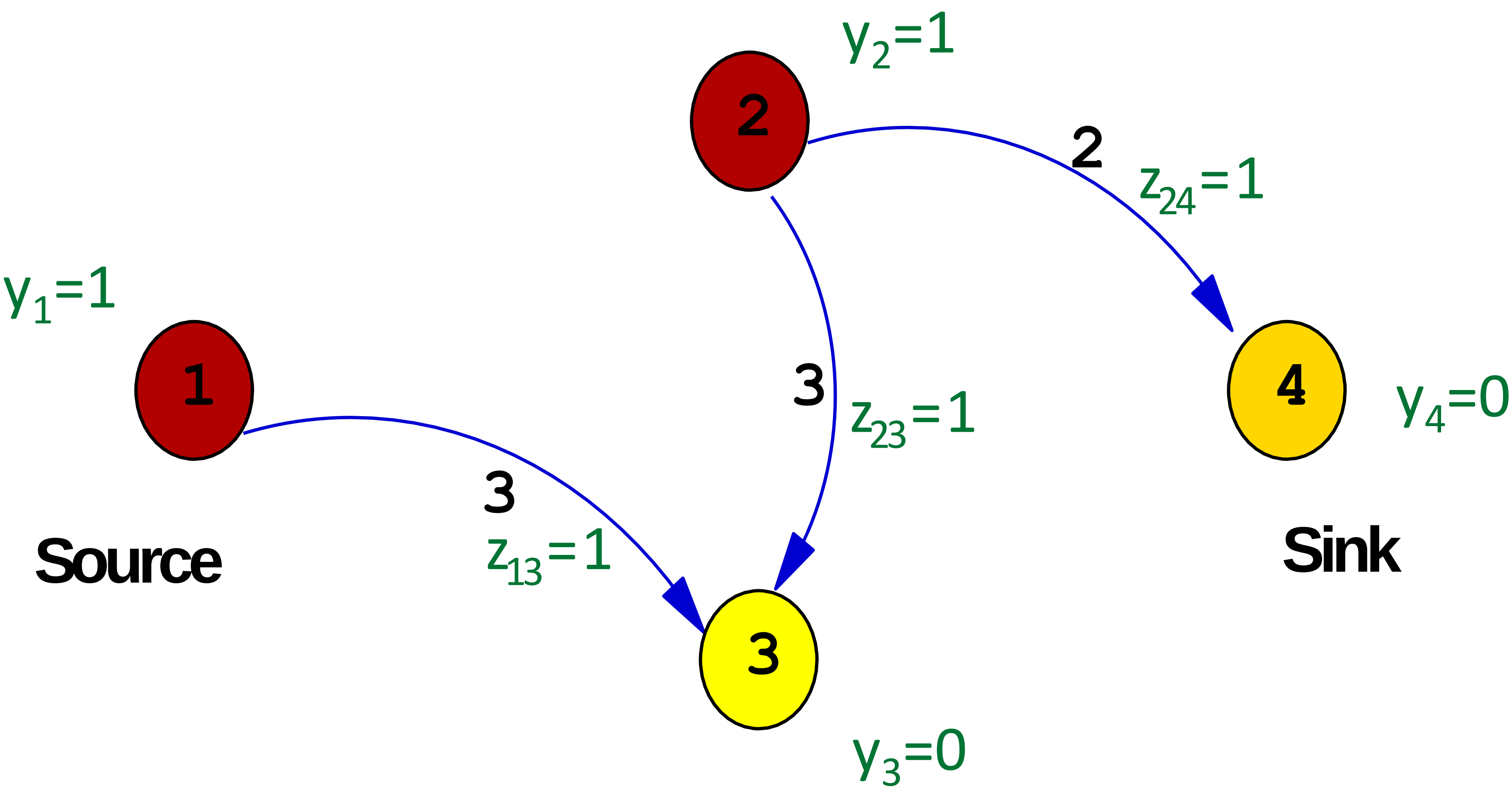
y_i: node potential value; wlog set $y_4 = 0$ so that $y_1 = 1$ and at optimality for all other y_i:

$$y_i = \begin{cases} 1 & \text{if } i \text{ is on the source side} \\ 0 & \text{if } i \text{ is not in the source side} \end{cases}$$

and

$$z_{ij} = \begin{cases} 1, & \text{if } y_i = 1 \text{ and } y_j = 0 \\ 0 & \text{otherwise} \end{cases}$$

The Min-Cut Solution: Min-Cut Value=8



All other $z_{ij}=0$

The Dual of the Information Market Problem

The ith order is given as triple $(\mathbf{a}_i \in R^m, \pi_i \in R_+, q_i \in R_+)$:

$$\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{im})$$

is the betting indication row vector where each component is either 1 or 0, where 1 is winning state and 0 is non-winning state;

π_i is the bidding price for one share of such a contract, and

q_i is the maximum number of shares the bidder like to own.

A contract /share on an order is a paper agreement so that on maturity it is worth a notional \$1 dollar if the order includes the winning state and worth \$0 otherwise.

Let x_i be the number of units awarded to the ith order.

A Risk-Free Mechanism of Market Maker

Corresponding Dual Variables

$$
\begin{array}{llll}
\max & \pi^T \mathbf{x} - x_{n+1} & & \\
\text{s.t.} & A^T \mathbf{x} - \mathbf{1} \cdot x_{n+1} \le \mathbf{0} & & \boldsymbol{p} \\
& \mathbf{x} \le \mathbf{q} & & \boldsymbol{s} \\
& \mathbf{x} \ge \mathbf{0} & & \\
& x_{n+1} \text{ free} & &
\end{array}
$$

where $\mathbf{1}$ is the vector of all ones.

$\pi^T \mathbf{x}$: the revenue amount can be collected.

x_{n+1} : the worst-case cost (amount need to pay to the winners).

The Dual: Regression with “Under-Bid” Filtering

$$
\begin{array}{lll}
\min & \boldsymbol{q}^T \boldsymbol{s} & \\
\text{s.t.} & A\boldsymbol{p} + \boldsymbol{s} & \geq \boldsymbol{\pi}, \\
 & -\boldsymbol{1}^T \boldsymbol{p} & = -1, \\
 & (\boldsymbol{p}, \boldsymbol{s}) & \geq 0.
\end{array}
$$

$\boldsymbol{p}_j$: the shadow/dual price of state j;

$\boldsymbol{a}_i\boldsymbol{p}$: the ith order unit cost at prices $\boldsymbol{p}$;

$\boldsymbol{s}_j$: the unit profit from the jth order ($\boldsymbol{s}=\max\{\boldsymbol{0}, \boldsymbol{\pi}-A\boldsymbol{p}\}$)

The dual problem is to minimize the total “Regression Loss” collected from the (competitive or high-bid) orders, $\boldsymbol{q}^T \boldsymbol{s}$.

ReLu-Regression for Probability Distribution/Information

$$\begin{array}{lll} \min & \boldsymbol{q}^T \max\{\boldsymbol{0}, \boldsymbol{\pi}\text{-}A\boldsymbol{p}\} & \\ \text{s.t.} & \boldsymbol{1}^T \boldsymbol{p} & = 1, \\ & \boldsymbol{p} & \geq 0 \end{array}$$

$\boldsymbol{p}_j$: the shadow-price/probability estimation of state j;

$\boldsymbol{a}_i\boldsymbol{p}$: the ith order unit cost at prices $\boldsymbol{p}$;

$\boldsymbol{\pi}_i$: the ith order bidding price;

$\boldsymbol{q}_i$: the ith order quantity limit;

The dual problem is to minimize the total weighted discrepancy among the competitive bidders such that all winners' betting beliefs $\boldsymbol{\pi}$ are fully utilized, while under-bidders (outliers) would be automatically removed from the estimation.

The World Cup Betting Example

Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00