The Transportation Simplex Method

Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

https://canvas.stanford.edu/courses/179677

Read Chapters 2.3, 4.1-4.2, 4.5

Recall the Simplex Method

1. Initialize with a minimization problem in the Tableau form with respect to a basic index set B. Let N denote the complementary index set.
2. Test for termination. First find (Dantzig rule): $r_e = \min_{j \in N} \{r_j\}$.

    If $r_e \geq 0$, stop. The solution is optimal. Otherwise determine whether the column associated with x_e contains a positive entry. If not, the objective function is unbounded below. Terminate.

3. Ratio Test. Execute the RT to determine the maximum increase, the pivot row o and the pivot element.
4. Pivot step. Pivoting and updating the Tableau, the indexes of B. Return to Step 1.

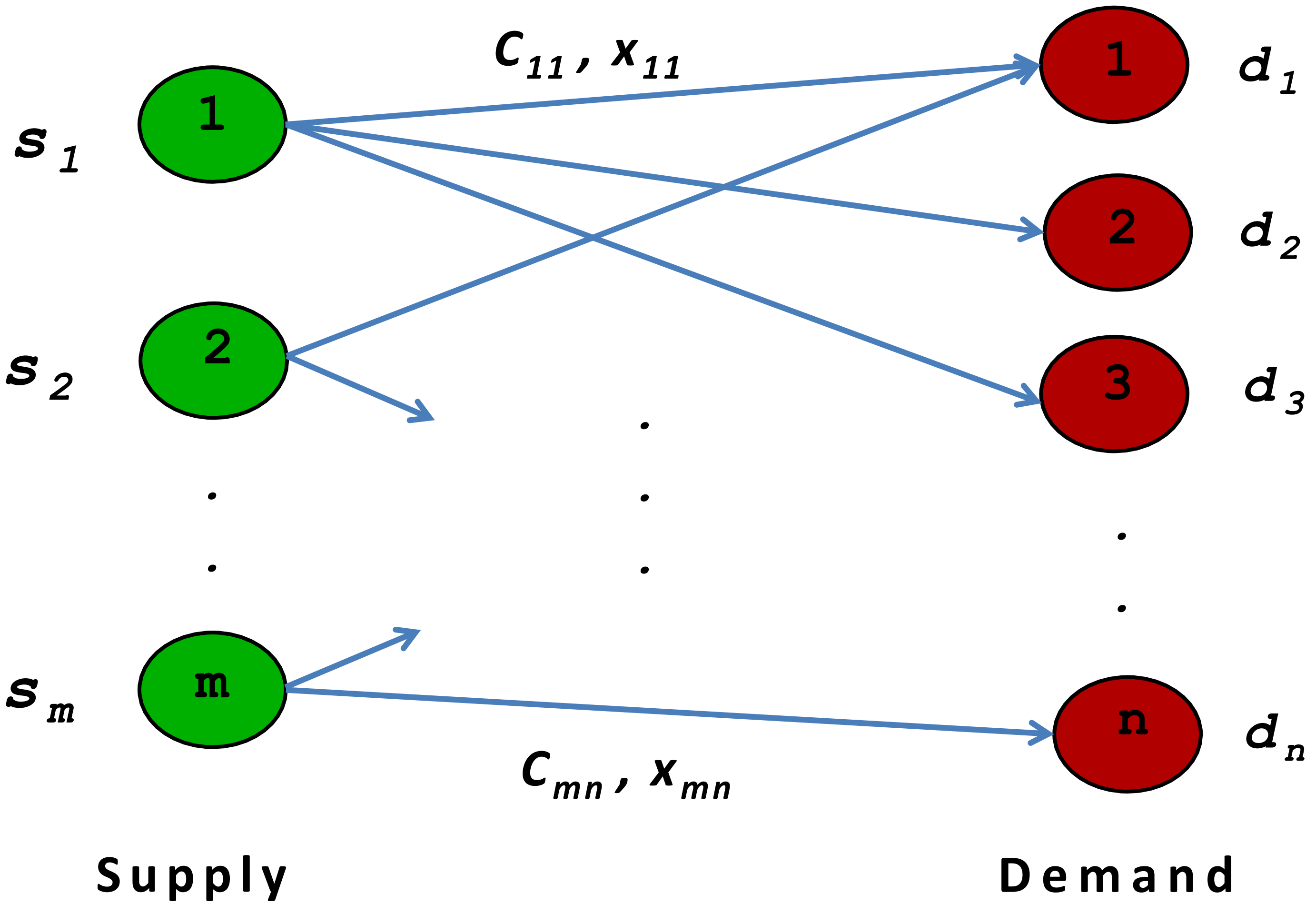
The Transportation Simplex Method

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = s_i, \quad \forall\, i = 1,\dots,m \\
& \sum_{i=1}^{m} x_{ij} = d_j, \quad \forall\, j = 1,\dots,n \\
& x_{ij} \ge 0, \qquad \forall\, i, j
\end{aligned}$$

Assume that the total supply equal the total demand. Thus, exactly one equality constraint is redundant.

At each step the simplex method attempts to send units along a route that is unused (non-basic) in the current BFS, while eliminating one of the routes that is currently being used (basic).

Transportation and Supply Chain Network



The Transportation Data Table

	1	2	3	4	Supply
1	12	13	4	6	500
2	6	4	10	11	700
3	10	9	12	4	800
Demand	400	900	200	500	2000

Transportation Simplex Method: Phase I

1. Start with the cell in the northwest corner cell
2. Allocate as many units as possible, consistent with the available supply and demand.
3. Move one cell to right if there is remaining supply; otherwise, move one cell down.
4. goto Step 2.

				500
				700
				800
400	900	200	500	

North-West Corner Method: Compute a BFS

400				100
				700
				800
0	900	200	500	

North-West Corner Method: Compute a BFS

400	100			0
				700
				800
0	800	200	500	

North-West Corner Method: Compute a BFS

400	100			0
	700			0
				800
0	100	200	500	

North-West Corner Method: Compute a BFS

400	100			0
	700			0
	100			700
0	0	200	500	

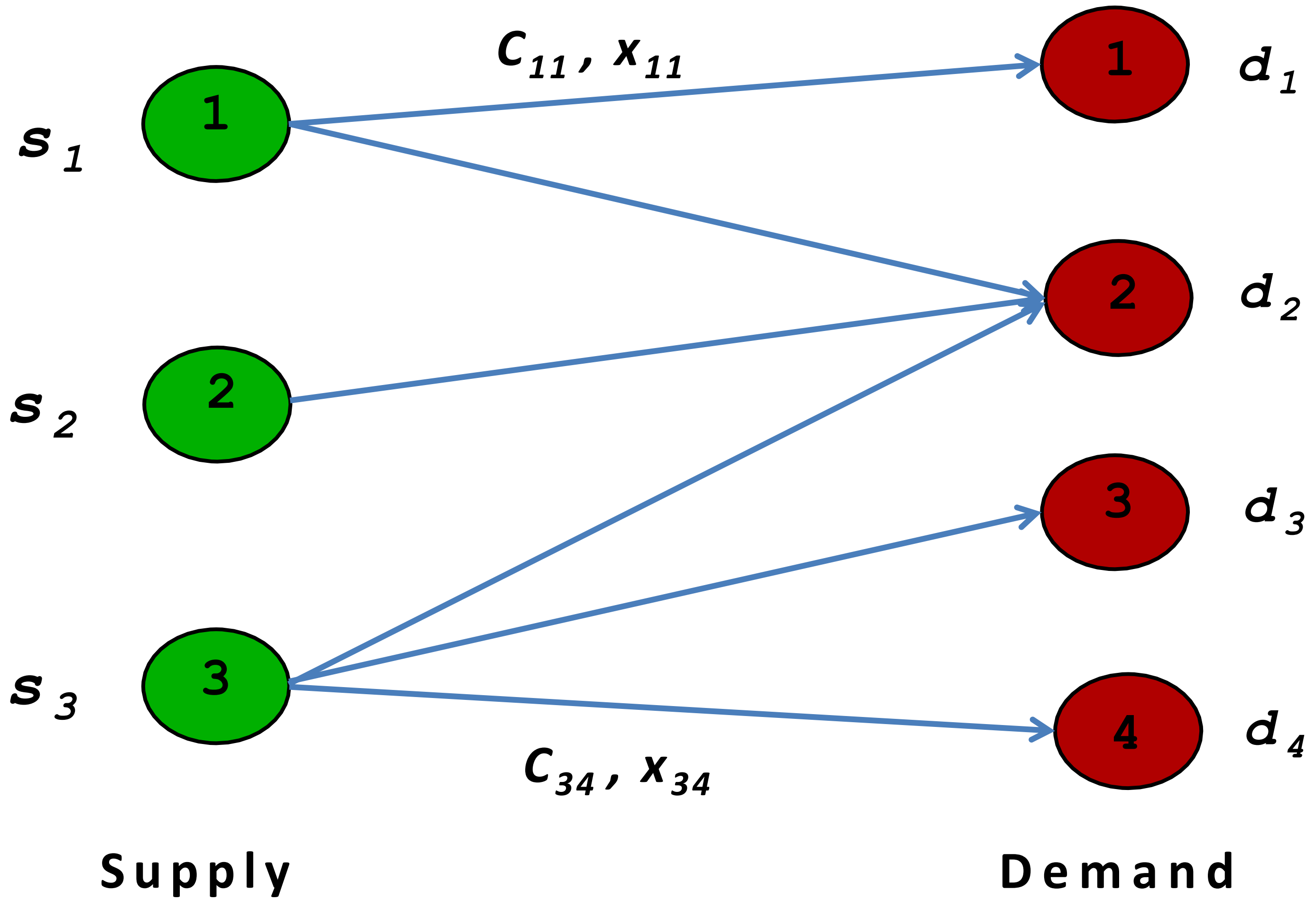
North-West Corner Method: Compute a BFS

400	100			0
	700			0
	100	200		500
0	0	0	500	

North-West Corner Method: Compute a BFS

400	100			0
	700			0
	100	200	500	0
0	0	0	0	

A BFS as a “Tree” Structure in the Network



(Tailored) Transportation Simplex Method: Phase II

1. Determine the shadow prices (for each supply side u_i and each demand side v_j) from every USED cell (basic variable)

$$\mathbf{y}^T = \mathbf{c}^T{}_B(A_B)^{-1} \Rightarrow \mathbf{y}^T A_B = \mathbf{c}^T{}_B \Rightarrow u_i + v_j = c_{ij}$$

One can always set $v_n = 0$ by viewing the last demand constraint redundant. Then do back-substitution…

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=$
	100 9	200 12	500 4	0 $u_3=$
0 $v_1=$	0 $v_2=$	0 $v_3=$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=$	0 $v_3=$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=$	0 $v_3=8$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=5$	0 $v_3=8$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=-1$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=5$	0 $v_3=8$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 u_1=8
	700 4			0 u_2=-1
	100 9	200 12	500 4	0 u_3=4
0 v_1=	0 v_2=5	0 v_3=8	0 v_4=0	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=8$
	700 4			0 $u_2=-1$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=4$	0 $v_2=5$	0 $v_3=8$	0 $v_4=0$	

Transportation Simplex Method: Phase II

1. Determine the shadow prices (for each supply side u_i and each demand side v_j) from every USED cell (basic variable)

$$\mathbf{y}^{\mathbf{T}} = \mathbf{c}^{\mathbf{T}}_B(A_B)^{-1} \Rightarrow \mathbf{y}^{\mathbf{T}} A_B = \mathbf{c}^{\mathbf{T}}_B \Rightarrow u_i + v_j = c_{ij}$$

One can always set $v_n = 0$ by viewing the last demand constraint redundant; then do back-substitution…

2. Calculate the reduced costs for the UNUSED cells (non-basic variable)

$$r_N = c^T_N - y^T A_N \Rightarrow r_{ij} = c_{ij} - u_i - v_j$$

If the reduced cost for every unused cell is nonnegative, then STOP: declare OPTIMAL

Step 2: Compute Reduced Costs

400 12	100 13	4	6	500 $u_1=8$
6	700 4	10	11	700 $u_2=-1$
10	100 9	200 12	500 4	800 $u_3=4$
400 $v_1=4$	900 $v_2=5$	200 $v_3=8$	500 $v_4=0$	2000

$$r_{ij} = c_{ij} - u_i - v_j$$

Step 2: Compute Reduced Costs

<table>
<tr><td>400
12 0</td><td>100
13 0</td><td>4 -12</td><td>6 -2</td><td>500
u_1=8</td></tr>
<tr><td>6 3</td><td>700
4 0</td><td>10 3</td><td>11 12</td><td>700
u_2=-1</td></tr>
<tr><td>10 2</td><td>100
9 0</td><td>200
12 0</td><td>500
4 0</td><td>800
u_3=4</td></tr>
<tr><td>400
v_1=4</td><td>900
v_2=5</td><td>200
v_3=8</td><td>500
v_4=0</td><td>2000</td></tr>
</table>

Reduced costs are computed in RED

Transportation Simplex Method: Phase II

1. Determine the shadow prices (for each supply side u_i and each demand side v_j) from every USED cell (basic variable)

$$\mathbf{y}^{\mathbf{T}} = \mathbf{c}^{\mathbf{T}}_B(A_B)^{-1} \Rightarrow \mathbf{y}^{\mathbf{T}} A_B = \mathbf{c}^{\mathbf{T}}_B \Rightarrow u_i + v_j = c_{ij}$$

One can always set $v_n = 0$ by viewing the last demand constraint redundant; then do back-substitution…

2. Calculate the reduced costs for the UNUSED cells (non-basic variable)

$$r_N = c^T_N - y^T A_N \Rightarrow r_{ij} = c_{ij} - u_i - v_j$$

If the reduced cost for every unused cell is nonnegative, then STOP: declare OPTIMAL

3. Select an unused cell with the most negative reduced cost as in-coming. Using a chain-reaction-cycle, determine the max units (α) that can be allocated to the in-coming cell and adjust the allocation appropriately. Update the values of the new set of USED (basic) cells (a new BFS).

Step 3: Chain Reaction Cycle

400	100	+α		0
	700			0
	100	200 -α	500	0
0	0	0	0	

Step 3: Chain Reaction Cycle

400	100	+α		0
	700			0
	100 +α	200 -α	500	0
0	0	0	0	

Step 3: Chain Reaction Cycle

400	100 -α	+α		0
	700			0
	100 +α	200 -α	500	0
0	0	0	0	

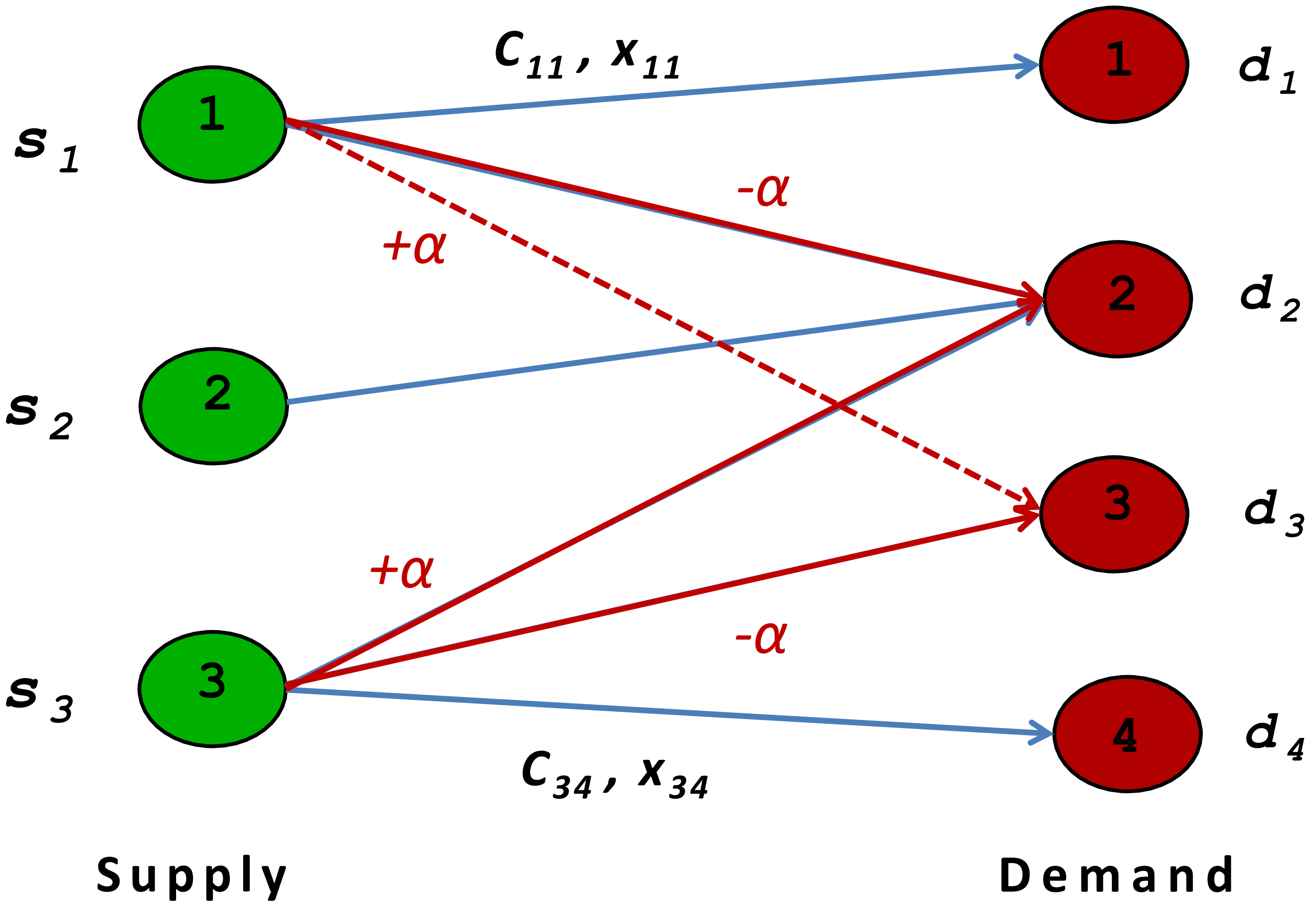
α = 100

Step 3: Chain Reaction Cycle

400	100 13 -α	4 +α		0
	700			0
	100 9 +α	200 12 -α	500	0
0	0	0	0	

α = 100, and the cost is reduced by 1200

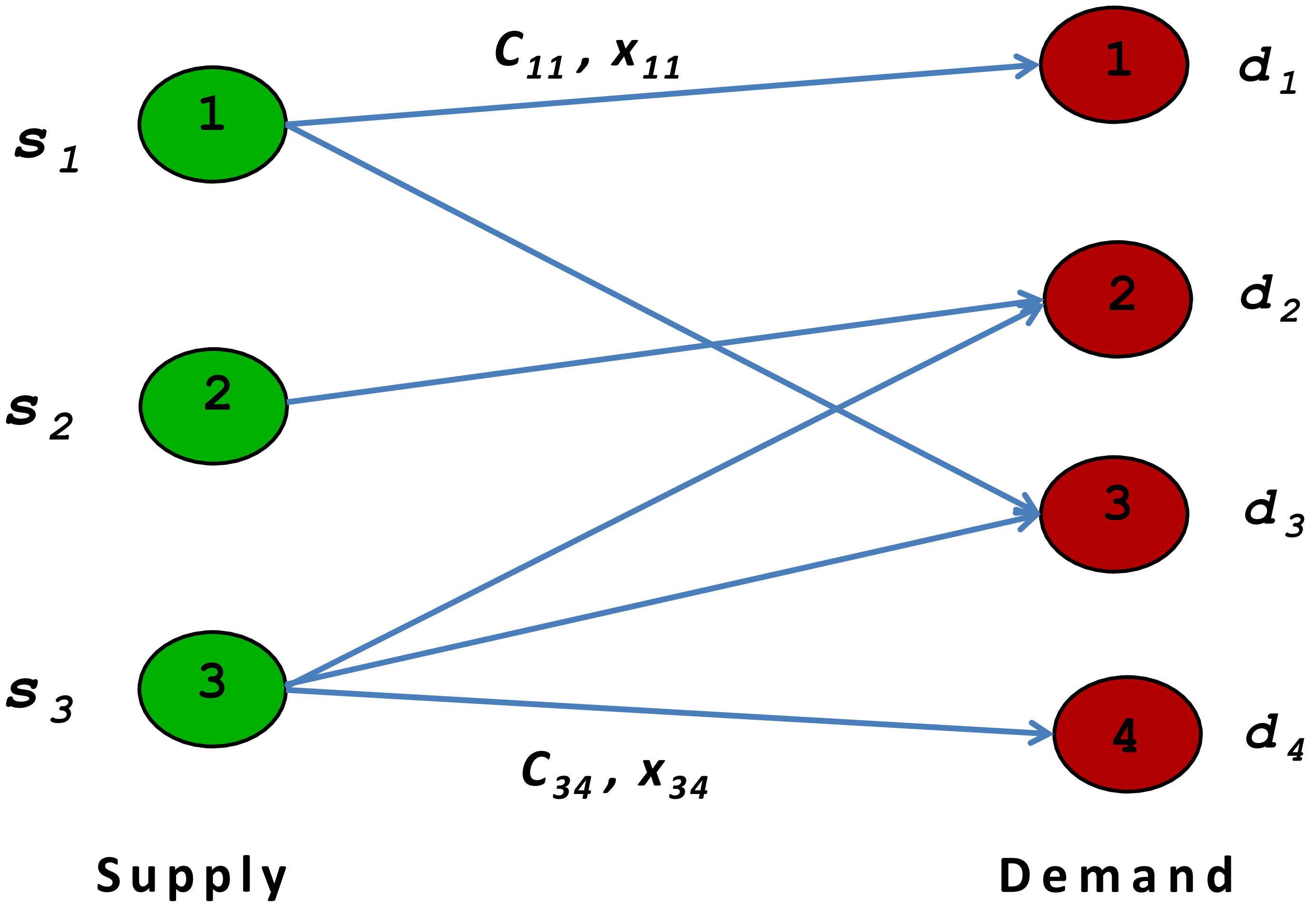
Find the Cycle on the "Tree" Structure



Step 3: Update to the New BFS

400		100		0
	700			0
	200	100	500	0
0	0	0	0	

A New “Tree” Structure in the Network: Repeat the Procedure



Transportation Simplex Method: Phase II

1. Determine the shadow prices (for each supply side u_i and each demand side v_j) from every USED cell (basic variable)

$$\mathbf{y}^\mathbf{T} = \mathbf{c}^\mathbf{T}_B(A_B)^{-1} \Rightarrow \mathbf{y}^\mathbf{T} A_B = \mathbf{c}^\mathbf{T}_B \Rightarrow u_i + v_j = c_{ij}$$

One can always set $v_n = 0$ by viewing the last demand constraint redundant; then do back-substitution…

2. Calculate the reduced costs for the UNUSED cells (non-basic variable)

$$r_N = c^T_N - y^T A_N \Rightarrow r_{ij} = c_{ij} - u_i - v_j$$

   If the reduced cost for every unused cell is nonnegative, then STOP: declare OPTIMAL

3. Select an unused cell with the most negative reduced cost as in-coming. Using the minRT, chain-reaction-cycle, determine the max units (α) that can be allocated to the in-coming cell and adjust the allocation appropriately. Update the values of the new set of USED (basic) cells (a new BFS).

Go to Step 1