Linear Programming Standard Equality Form and Solution Properties

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https://web.stanford.edu/class/msande211x/handout.shtml

Chapters 2.3-2.5, 4.1-4.2, 4.5

LP in Standard (**Equality**) Form

$$\min \quad c^T x = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad a_1 x = \sum_{j=1}^{n} a_{1j} x_j = b_1$$

$$a_2 x = \sum_{j=1}^{n} a_{2j} x_j = b_2$$

$$\ldots$$

$$a_m x = \sum_{j=1}^{n} a_{mj} x_j = b_m$$

$$x \geq 0$$

⇒

$$\min \quad c^T x$$

$$\text{s.t.} \quad Ax = b,$$

$$x \geq 0$$

Reduction to Standard Form

- max $\boldsymbol{c}^T\boldsymbol{x}$ to min $-\boldsymbol{c}^T\boldsymbol{x}$
- Eliminating "free" variables: substitute with the difference of two nonnegative variables

$$x := x^{I} - x^{II}, \quad (x^{I}, x^{II}) \geq 0.$$

- Eliminating inequalities: add a slack variable

$$\mathbf{a}^T\mathbf{x} \leq b = \Rightarrow \mathbf{a}^T\mathbf{x} + s = b, \quad s \geq 0$$

$$\mathbf{a}^T\mathbf{x} \geq b = \Rightarrow \mathbf{a}^T\mathbf{x} - s = b, \quad s \geq 0$$

Reduction of the Production Problem

$$\begin{aligned}
\max \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & x_1 \le 1 \\
& x_2 \le 1 \\
& x_1 + x_2 \le 1.5 \\
& x_1 \ge 0 \\
& x_2 \ge 0
\end{aligned}$$

⇒

$$\begin{aligned}
\min \quad & -x_1 - 2x_2 \\
\text{s.t.} \quad & x_1 + x_3 = 1 \\
& x_2 + x_4 = 1 \\
& x_1 + x_2 + x_5 = 1.5 \\
& (x_1, x_2, x_3, x_4, x_5) \ge 0
\end{aligned}$$

x_3, x_4,and x_5 are called slack variables

We know how to identify corners/extreme-points of the LP feasible region defined all by linear inequalities. What about corners in this LP standard equality form?

How to Identify Corners in LP Equality Form
Basic and Basic Feasible Solution

In the LP standard form, select m linearly independent columns, denoted by the variable index set B, from A. Solve

$$A\mathbf{x} = \mathbf{b} \quad \Longleftrightarrow \quad A_B\mathbf{x}_B = \mathbf{b},\ \mathbf{x}_N = \mathbf{0}$$

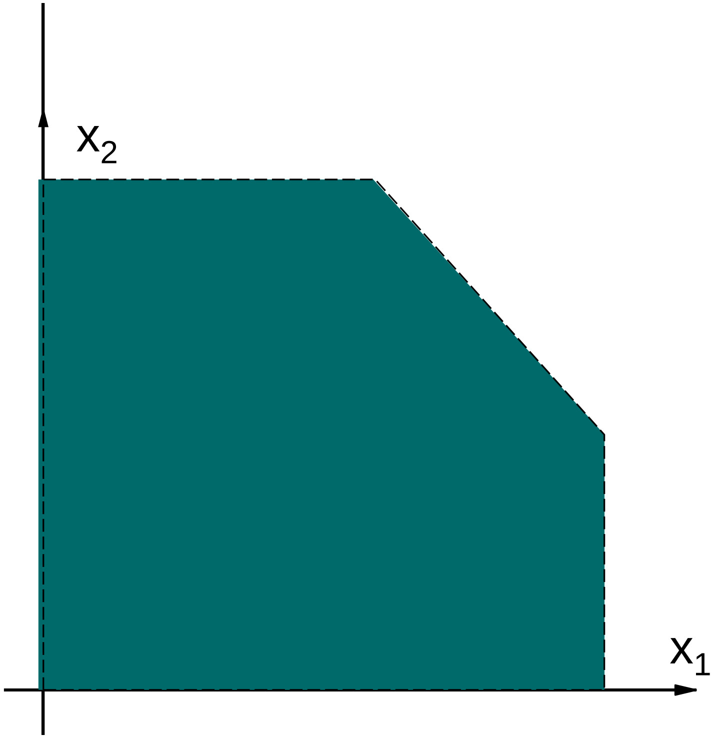
for the dimension-m vector $\mathbf{x}_B$. By setting the variables, $\mathbf{x}_N$, of $\mathbf{x}$ corresponding to the remaining columns of A equal to zero, we obtain a solution $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

Then, $\mathbf{x}$ is said to be a basic solution to (LP) with respect to the basic variable set B. The variables in $\mathbf{x}_B$ are called basic variables, those in $\mathbf{x}_N$ are nonbasic variables, and A_B is called a basis.

If a basic solution $\mathbf{x}_B \geq \mathbf{0}$, then $\mathbf{x}$ is called a basic feasible solution, or BFS. Note that A_B and $\mathbf{x}_B$ follow the same index order in B.

Two BFS are adjacent if they differ by exactly one basic variable.

BS of the Production Problem in Equality Form



$$
\begin{array}{llllll}
x_1 & & +x_3 & & & = 1 \\
& x_2 & & +x_4 & & = 1 \\
x_1 & +x_2 & & & +x_5 & = 1.5 \\
(x_1, & x_2, & x_3, & x_4, & x_5) & \geq 0
\end{array}
$$

Basis	3,4,5	1,4,5	3,4,1	3,2,5	3,4,2	1,2,3	1,2,4	1,2,5
Feasible?	√	√		√		√	√	
x_1 , x_2	0, 0	1, 0	1.5, 0	0, 1	0, 1.5	.5, 1	1, .5	1, 1

BFS and Corner Point Equivalence Theorem

Theorem *Consider the feasible region in the standard LP form. Then, a basic feasible solution and a corner (extreme) point are equivalent; the formal is algebraic and the latter is geometric. Moreover, Two corners are neighboring if exact one variable difference in basis*

- Feasible directions of an BFS: an increasing direction of the nonbasic variables (they equal 0 right now).
- Extreme feasible direction: the increasing direction of a nonbasic variable x_j: $\mathbf{x}_B = (A_B)^{-1}\mathbf{b} - (A_B)^{-1}\mathbf{a}_j x_j$
- Optimality test: No improving (extreme) feasible direction exists

Feasible Directions at a BFS and Optimality Test

- Recall at a BFS: $A_B x_B + A_N x_N = b$, and $x_B \geq 0$ and $x_N = 0$.

Thus we can express x_B in terms of x_N,

$$x_B = (A_B)^{-1}b - (A_B)^{-1}A_N x_N .$$

Reduced Objective

Then, $c^T x = c^T_B x_B + c^T_N x_N = (c^T_N - c^T_B (A_B)^{-1}A_N)\, x_N + c^T_B (A_B)^{-1}b$

- Note that increase any one variable of x_N is an extreme feasible direction. Thus, for the BFS to be optimal, any (extreme) feasible direction must be an ascent direction, or

$$(c^T_N - c^T_B (A_B)^{-1}A_N) \geq 0$$

is necessary and sufficient for the current BFS being optimal!

- This vector is called the reduced cost coefficient vector or reduced gradient vector from the current BFS. Note that reduced cost coefficients for basic variables are all zeros.

The Simplex Method: Shadow-Price and Reduced Cost Vectors

We first introduce and compute an intermediate shadow-price/multiplier vector:

$$y^T = c^T_B (A_B)^{-1} \text{ , or } y^T A_B = c^T_B,$$

by solving a system of linear equations.

Then we compute reduced cost $r^T = c^T - y^T A$, where r_N is the reduced cost vector for nonbasic variables (and $r_B = 0$ always).

If one of r_N is negative, then an improving (extreme) feasible direction is find by increasing the corresponding nonbasic variable value.

In the LP production example, suppose the basic variable set *B = {3, 4, 5}* .

$$
\begin{array}{llllllll}
\min & -x_1 & -2x_2 & & & & \\
\text{s.t.} & x_1 & & +x_3 & & & = 1 \\
 & & x_2 & & +x_4 & & = 1 \\
 & x_1 & +x_2 & & & +x_5 & = 1.5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
$$

$$
c_N = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, c_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, A_B = I, A_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},
$$

$$
A_B^{-1} = I, y^T = (0 \quad 0 \quad 0), \quad r_N^T = (-1 \quad -2).
$$

Thus, increasing either x_1 and x_2 is a feasible and improving direction and the variable is called the incoming basic variable…

In the LP production example, suppose the basic variable set $B = \{1, 2, 3\}$.

$$
\begin{array}{lrrrrrl}
\min & -x_1 & -2x_2 & & & & \\
\text{s.t.} & x_1 & & +x_3 & & & = 1 \\
 & & x_2 & & +x_4 & & = 1 \\
 & x_1 & +x_2 & & & +x_5 & = 1.5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
$$

$$
c_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, c_B = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}, A_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, A_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
A_B^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad y^T = (0 \quad -1 \quad -1), \quad r_N^T = (1 \quad 1).
$$

Thus, this BFS is optimal

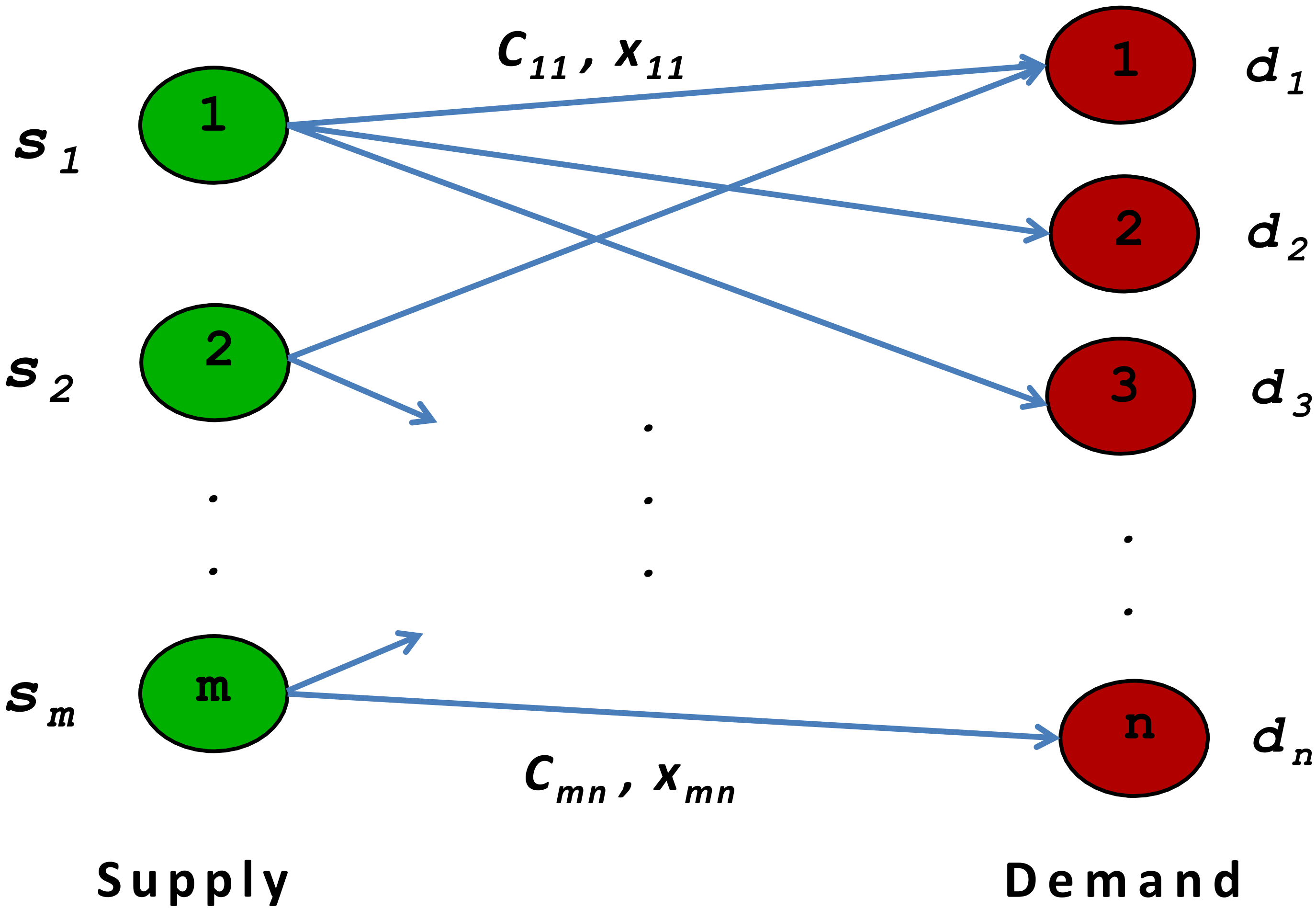
The Transportation Simplex Method

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} = s_i, & \forall\, i = 1,\ldots,m \\
& \sum_{i=1}^{m} x_{ij} = d_j, & \forall\, j = 1,\ldots,n \\
& x_{ij} \geq 0, & \forall\, i, j
\end{array}
$$

Assume that the total supply equal the total demand. Thus, exactly one equality constraint is redundant.

At each step the simplex method attempts to send units along a route that is unused (non-basic) in the current BFS, while eliminating one of the routes that is currently being used (basic).

Transportation and Supply Chain Network



The Transportation Data Table

	1	2	3	4	Supply
1	12	13	4	6	500
2	6	4	10	11	700
3	10	9	12	4	800
Demand	400	900	200	500	2000

Transportation Simplex Method: Phase I

1. Start with the cell in the northwest corner cell
2. Allocate as many units as possible, consistent with the available supply and demand.
3. Move one cell to right if there is remaining supply; otherwise, move one cell down.
4. goto Step 2.

				500
				700
				800
400	900	200	500	

North-West Corner Method: Compute a BFS

400				100
				700
				800
0	900	200	500	

North-West Corner Method: Compute a BFS

400	100			0
				700
				800
0	800	200	500	

North-West Corner Method: Compute a BFS

400	100			0
	700			0
				800
0	100	200	500	

North-West Corner Method: Compute a BFS

400	100			0
	700			0
	100			700
0	0	200	500	

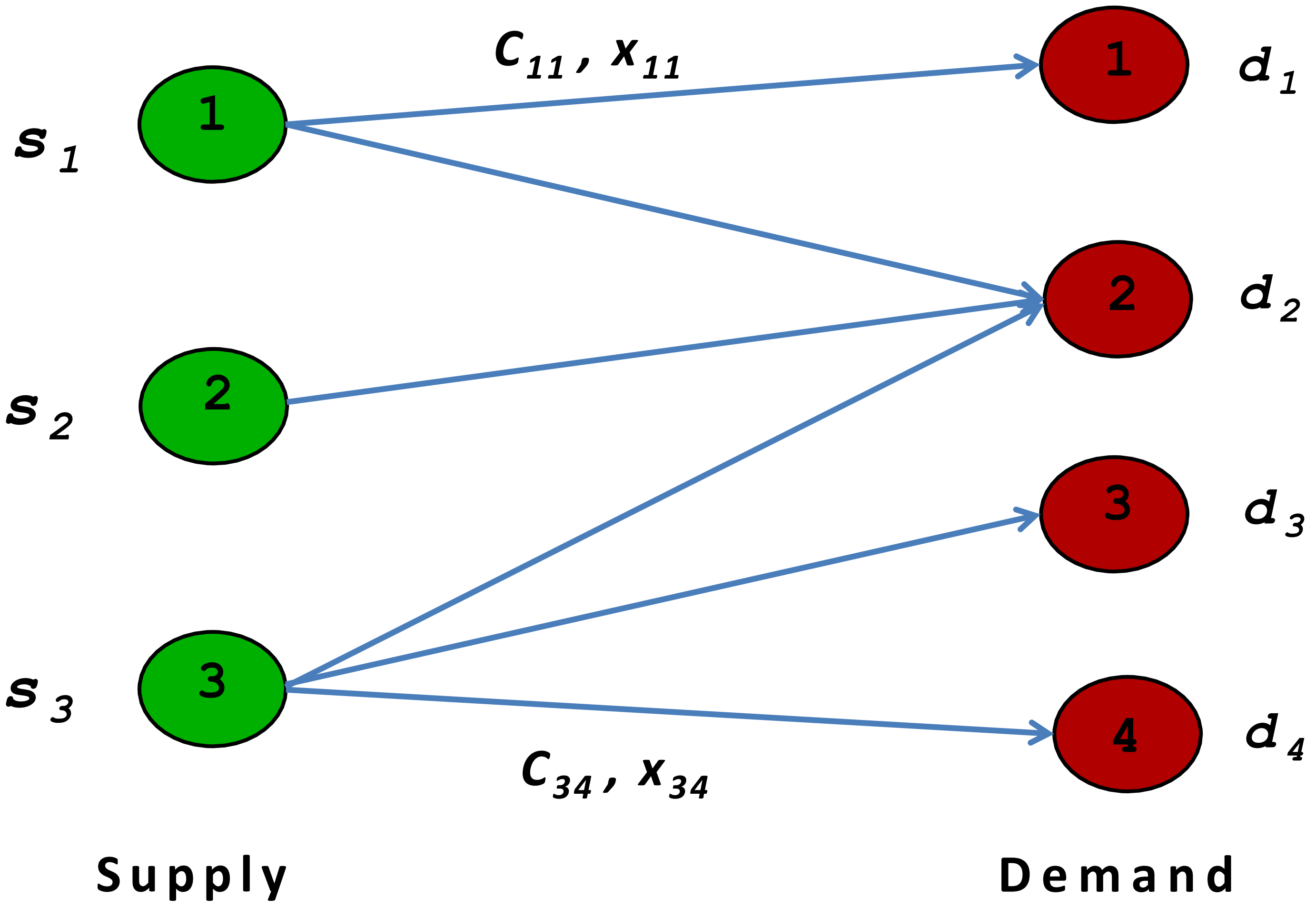
North-West Corner Method: Compute a BFS

400	100			0
	700			0
	100	200		500
0	0	0	500	

North-West Corner Method: Compute a BFS

400	100			0
	700			0
	100	200	500	0
0	0	0	0	

A BFS as a "Tree" Structure in the Network



(Tailored) Transportation Simplex Method: Phase II

1. Determine the shadow prices (for each supply side u_i and each demand side v_j) from every USED cell (basic variable)

$$\mathbf{y}^T = \mathbf{c}^T{}_B(A_B)^{-1} \Rightarrow \mathbf{y}^T A_B = \mathbf{c}^T{}_B \Rightarrow u_i + v_j = c_{ij}$$

One can always set $v_n = 0$ by viewing the last demand constraint redundant. Then do back-substitution…

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=$
	100 9	200 12	500 4	0 $u_3=$
0 $v_1=$	0 $v_2=$	0 $v_3=$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=$	0 $v_3=$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 u_1=
	700 4			0 u_2=
	100 9	200 12	500 4	0 u_3=4
0 v_1=	0 v_2=	0 v_3=8	0 v_4=0	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=5$	0 $v_3=8$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=$
	700 4			0 $u_2=-1$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=5$	0 $v_3=8$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=8$
	700 4			0 $u_2=-1$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=$	0 $v_2=5$	0 $v_3=8$	0 $v_4=0$	

Step 1: Compute Shadow Prices

400 12	100 13			0 $u_1=8$
	700 4			0 $u_2=-1$
	100 9	200 12	500 4	0 $u_3=4$
0 $v_1=4$	0 $v_2=5$	0 $v_3=8$	0 $v_4=0$	

Transportation Simplex Method: Phase II

1. Determine the shadow prices (for each supply side u_i and each demand side v_j) from every USED cell (basic variable)

$$\mathbf{y}^T = \mathbf{c}^T{}_B(A_B)^{-1} \Rightarrow \mathbf{y}^T A_B = \mathbf{c}^T{}_B \Rightarrow u_i + v_j = c_{ij}$$

One can always set $v_n = 0$ by viewing the last demand constraint redundant; then do back-substitution…

2. Calculate the reduced costs for the UNUSED cells (non-basic variable)

$$r_N = c^T{}_N - y^T A_N \Rightarrow r_{ij} = c_{ij} - u_i - v_j$$

If the reduced cost for every unused cell is nonnegative, then STOP: declare OPTIMAL

Step 2: Compute Reduced Costs

400 12	100 13	4	6	500 u_1=8
6	700 4	10	11	700 u_2=-1
10	100 9	200 12	500 4	800 u_3=4
400 v_1=4	900 v_2=5	200 v_3=8	500 v_4=0	2000

$$r_{ij} = c_{ij} - u_i - v_j$$

Step 2: Compute Reduced Costs

400 12 0	100 13 0	4 -12	6 -2	500 $u_1=8$
6 3	700 4 0	10 3	11 12	700 $u_2=-1$
10 2	100 9 0	200 12 0	500 4 0	800 $u_3=4$
400 $v_1=4$	900 $v_2=5$	200 $v_3=8$	500 $v_4=0$	2000

Reduced costs are computed in RED

Transportation Simplex Method: Phase II

1. Determine the shadow prices (for each supply side u_i and each demand side v_j) from every USED cell (basic variable)

$$\mathbf{y}^T = \mathbf{c}^T{}_B(A_B)^{-1} \Rightarrow \mathbf{y}^T A_B = \mathbf{c}^T{}_B \Rightarrow u_i + v_j = c_{ij}$$

One can always set $v_n = 0$ by viewing the last demand constraint redundant; then do back-substitution…

2. Calculate the reduced costs for the UNUSED cells (non-basic variable)

$$r_N = c^T{}_N - y^T A_N \Rightarrow r_{ij} = c_{ij} - u_i - v_j$$

If the reduced cost for every unused cell is nonnegative, then STOP: declare OPTIMAL

3. Select an unused cell with the most negative reduced cost as in-coming. Using a chain-reaction-cycle, determine the max units (α) that can be allocated to the in-coming cell and adjust the allocation appropriately. Update the values of the new set of USED (basic) cells (a new BFS).