More Linear Programming Examples II

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https://web.stanford.edu/class/msande211x/handout.shtml

Read Chapter 1.1, 1.2, 2.1, 2.2, Appendices A, B&D in Textbook

LP Example 8: Data Classification - Business or Personal?

Build a model that will predict a probability for each credit card transaction indicating whether the transaction is business or personal related.

- There is no training data where particular transactions are identified as being personal, we used personal remittances as the best proxy
- On the transaction side, we focused on the industry code of each transaction as a key initial differentiator between transactions
- Developed a LP model to establish probabilities for each industry code that indicate the likelihood that dollars spent in that code will be personal spending.

Transaction Types by Industrial Codes

Industry Code	Description
995	CLUB - WAREHOUSE
25	DEPARTMENT STORE - MASS MERCHANDISER
728	GASOLINE/OIL COMPANY - NATIONAL DEALER
729	GASOLINE/OIL COMPANY - INDEPENDENT DEALER
429	SHOP - HOME IMPROVEMENT
415	DEPARTMENT STORE - FULL SERVICE
87	INTERNET TRAVEL
504	SHOP - ELECTRONIC GOODS
616	COMMUNICATION - CABLE & BROADCAST SERVICES
215	AUTO SERVICES - MOTOR RELATED SERVICES/DEALER
404	AUTO SERVICES - AUTO SALES & SERVICE
443	SHOP - SPORTING GOODS
457	SHOP - CHEMIST/PHARMACY
522	SHOP - FURNITURE
463	SHOP - JEWELRY
757	ENTERTAINMENT - TICKET AGENT - COMPANY
407	SHOP - CLOTHING - FAMILY
680	SHOP - COMPUTER HARDWARE
465	SHOP - LIQUOR STORE
400	AUTO SERVICES - VEHICLE ACCESSORIES
416	DEPARTMENT STORE - SPECIALITY
428	SHOP - HOME FURNISHINGS
414	SHOP - CLOTHING - WOMEN'S
793	TRAVEL - TOUR OPERATOR GENERAL
412	SHOP - CLOTHING - MEN'S & WOMEN'S
787	TRAVEL - NON - `AGENT RETAILER
447	SHOP - SHOES - MEN'S ONLY
427	SHOP - HARDWARE/DO IT YOURSELF
554	MAIL ORDER SELF IMPROVEMENT/BUSINESS SEMINARS
603	SERVICES - BEAUTY SHOPS/BEAUTICIAN

Business Analytics

For each of the industry codes, the model will determine a probability which indicates the likelihood that a transaction was personal.

Each Column represents an Industry Code

Personal Remittances

				...					
Account	**1**	**2**	**3**	...	**n**			**Actual**	
1	**$156**	**$0**	**$87**		**$25**			**$200**	
2	**$200**	**$25**	**$0**		**$0**			**$195**	
…	**$0**	**$134**	**$35**		**$60**			**$210**	

Value of transactions in period

Model Example

For each of the industry codes, the model will determine a probability (in red) which indicates the likelihood that a transaction was personal. The goal is to minimize the sum of the squares of the differences (in blue).

Probability Personal

Each Column represents an Industry Code

Personal Remittances

	25%	10%	0%	…	5%				
Account	**1**	**2**	**3**	**…**	**n**		**Predicted**	**Actual**	**Difference**
1	$156	$0	$87		$25		$244	$200	$44
2	$200	$25	$0		$0		$200	$195	$5
…	$0	$134	$35		$60		$230	$210	$20

Value of transactions in period

LP Model: Constrained Linear Regression

Our model will determine the probability that a transaction from each industry code is personal in such a manner which will minimize the sum of the squared errors (between predicted personal remittances and actual personal remittances).

$$\text{Min} \quad \sum_i \left| \sum_j a_{ij} x_j - b_i \right|$$

$$\text{s.t.} \quad 0 \le x_j \le 1, \forall j.$$

- Let x_j be such a probability that a transaction is personal for industry code j
- $a_{i,j}$ – transaction amount for account i and industry code j
- b_i – amount paid by personal remit for account i
- $\sum_j a_{i,j} x_j$ – the expected personal expenses for account i
- We'd like to choose x_j such that $\sum_j a_{i,j} x_j$ matches b_i for ALL i

How to Linearize the Abs Function I

To dealing the abs function, we introduce auxiliary variables y_i

$$|z_i| = y_i, i = 1, \ldots, m.$$

Relax it to linear inequalities

$$-y_i \le z_i \le y_i, i = 1, \ldots, m.$$

If the sum of y_is is minimized, the equality must hold

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} y_i \\ \text{s.t.} \quad & -y_i \le \sum_j a_{ij}x_j - b_i \le y_i, \forall i \\ & 0 \le x_j \le 1, \forall j. \end{aligned}$$

This is an LP problem!

$$-y \le Ax - b \le y,$$
$$0 \le x \le 1$$

How to Linearize the Abs Function II

Introduce auxiliary variables y'_i and y''_i

$$z_i = y'_i - y''_i, y'_i \geq 0, y''_i \geq 0, i = 1, \ldots, m.$$

Relax it to linear inequalities

$$\min |z_i| \quad \Leftrightarrow \quad \min y'_i + y''_i$$

If the sum of y_is is minimized, the equality must hold

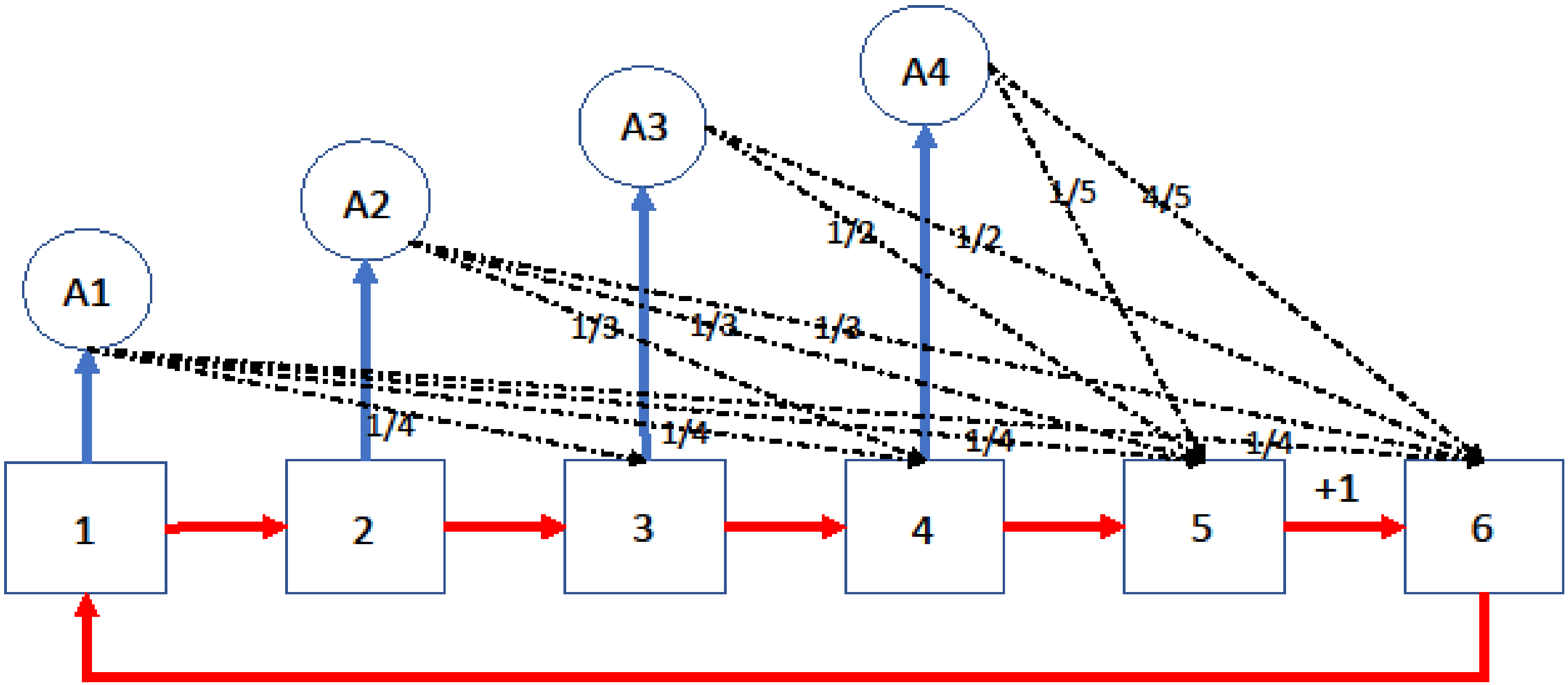
$$\begin{aligned} \min \quad & \sum_{i=1}^{m} (y'_i + y''_i) \\ \text{s.t.} \quad & Ax - b = y' - y'', \\ & 0 \leq x \leq 1, y' \geq 0, y'' \geq 0. \end{aligned}$$

This is an LP problem!

LP Example 9: Reinforcement Learning and Markov Decision Process

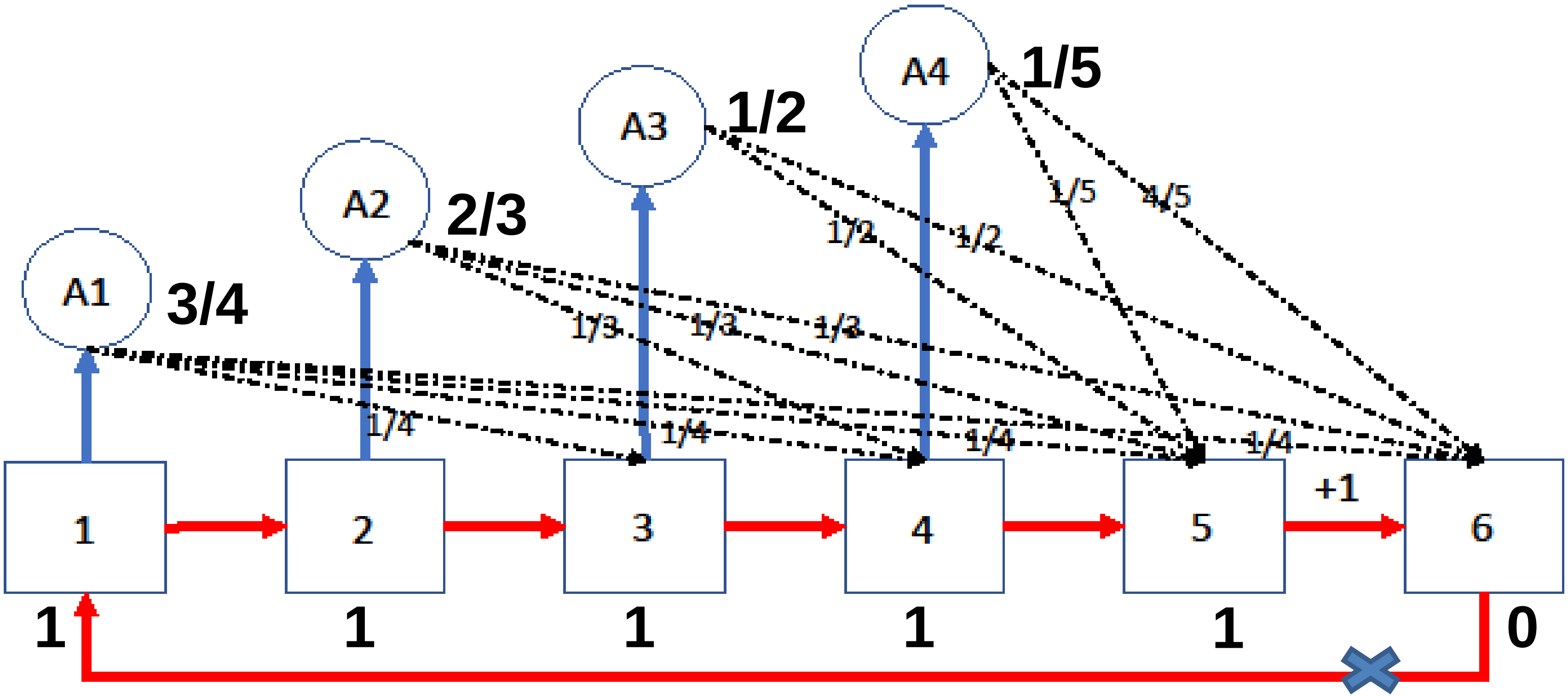
- Markov decision process provides a mathematical framework for modeling sequential decision-making in situations where outcomes are partly random and partly under the control of a decision maker, and it is called Reinforcement Learning lately.
- MDPs are useful for studying a wide range of optimization problems solved via dynamic programming, where it was known at least as early as the 1950s (cf. Shapley 1953, Bellman 1957).
- Modern applications include dynamic planning, social networking, and almost all other dynamic/sequential-decision-making problems in Mathematical, Physical, Management, Economics, and Social Sciences.
- MDP is characterized by States and Actions; and at each time step, the process is in a state and the decision maker chooses an action to optimize a long-term goal.

A Simple RL/MDP Problem: Maze Run



Each state *i* (in Square) is equipped with a set of actions A_i , and they are colored in red (status quo move), blue (shortcut move); and each of them incurs an immediate cost c_j. In this example, all actions have zero cost except the one from the state 4 (trap) to the final termination state 5 (Exit state which goes back to itself). Each action is associated with transition probability node (circle) with distribution vector P_j to all states.

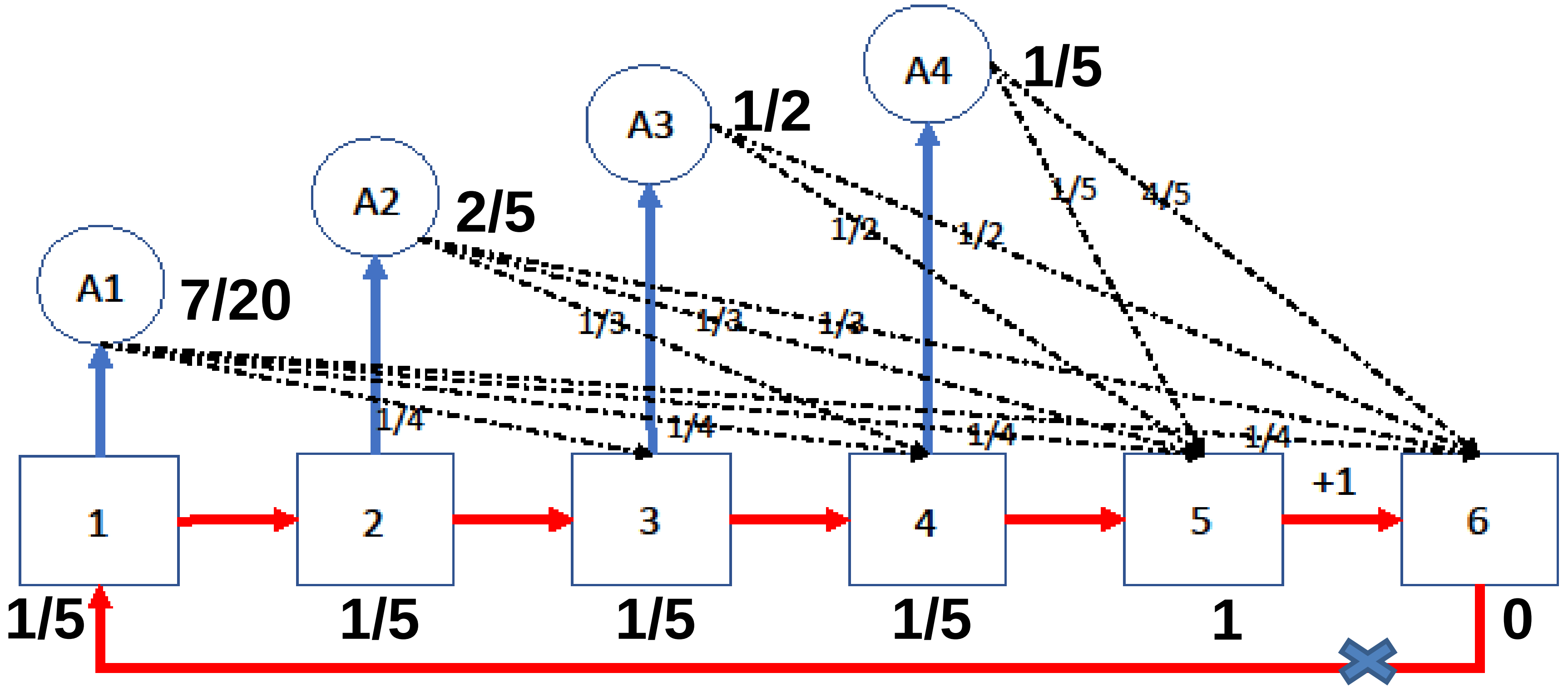
Cost-to-Go values of a Policy



A policy is a set of actions taken in each State at anytime, and it defines an expected Cost-to-Go value for each State (the overall present cost if starting from this very state). Assuming there is no discount and the current policy takes all-red actions, the corresponding expected cost-to-go state-values would be given above, together with expected values for blue-actions.

Clearly, this policy is not optimal…

Cost-to-Go values of another Policy

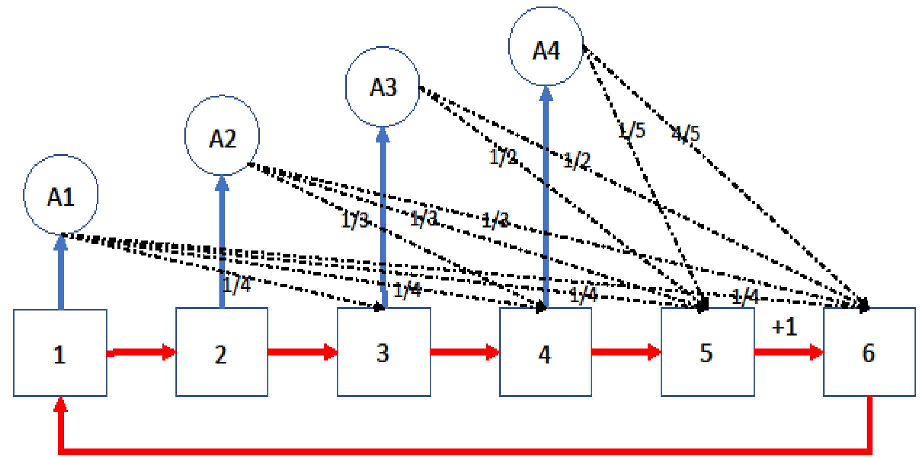


If the current policy is taking (red, red, red, blue, red) actions, the corresponding expected cost-to-go state-values would be given above, together with expected values for other actions. This policy is optimal.

An optimal policy is a policy that for each state there is no action-switch that results in a lower cost.

Cost-to-Go values of the Maze Run

- y_i: the expected overall present cost if stating from State i.
- State 5 is a trap
- State 6 is the exit state
- Each other state has two options:
  Go directly to the next state or
  a short-cut go to other states
  with uncertainties



- The cost-to-go values of the optimal policy with discount factor ϒ for this simple example should meet the following conditions

$$y_6= 0+ \gamma y_1 , \quad y_5=1+ \gamma y_6$$

$$y_4=\min\{0+ \gamma y_5 , 0+\gamma(0.2y_5+0.8y_6\},$$

$$y_3=\min\{0+ \gamma y_4 , 0+\gamma(0.5y_5+0.5y_6)\}$$

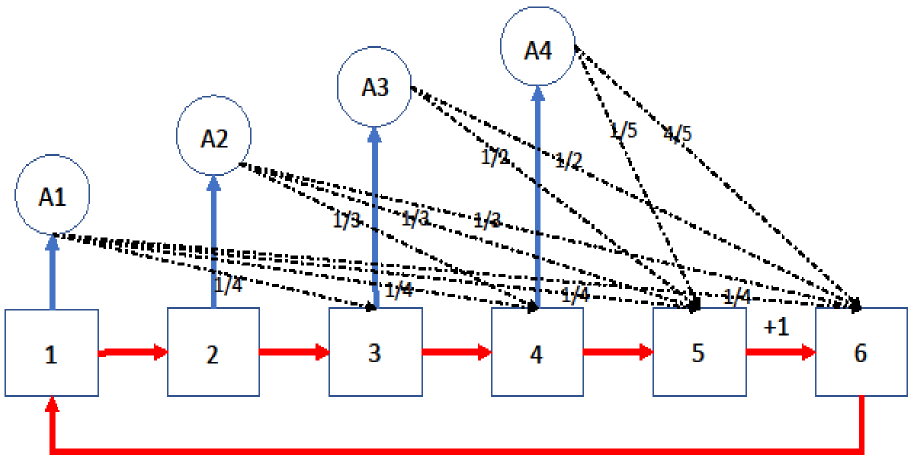
$$y_2=\min\{0+ \gamma y_3 , 0+\gamma(0.33y_4+0.33y_5+0.33y_6)\}$$

$$y_1=\min\{0+ \gamma y_2, 0+\gamma(0.25y_3+0.25y_4+0.25y_5+0.25y_6)\}$$

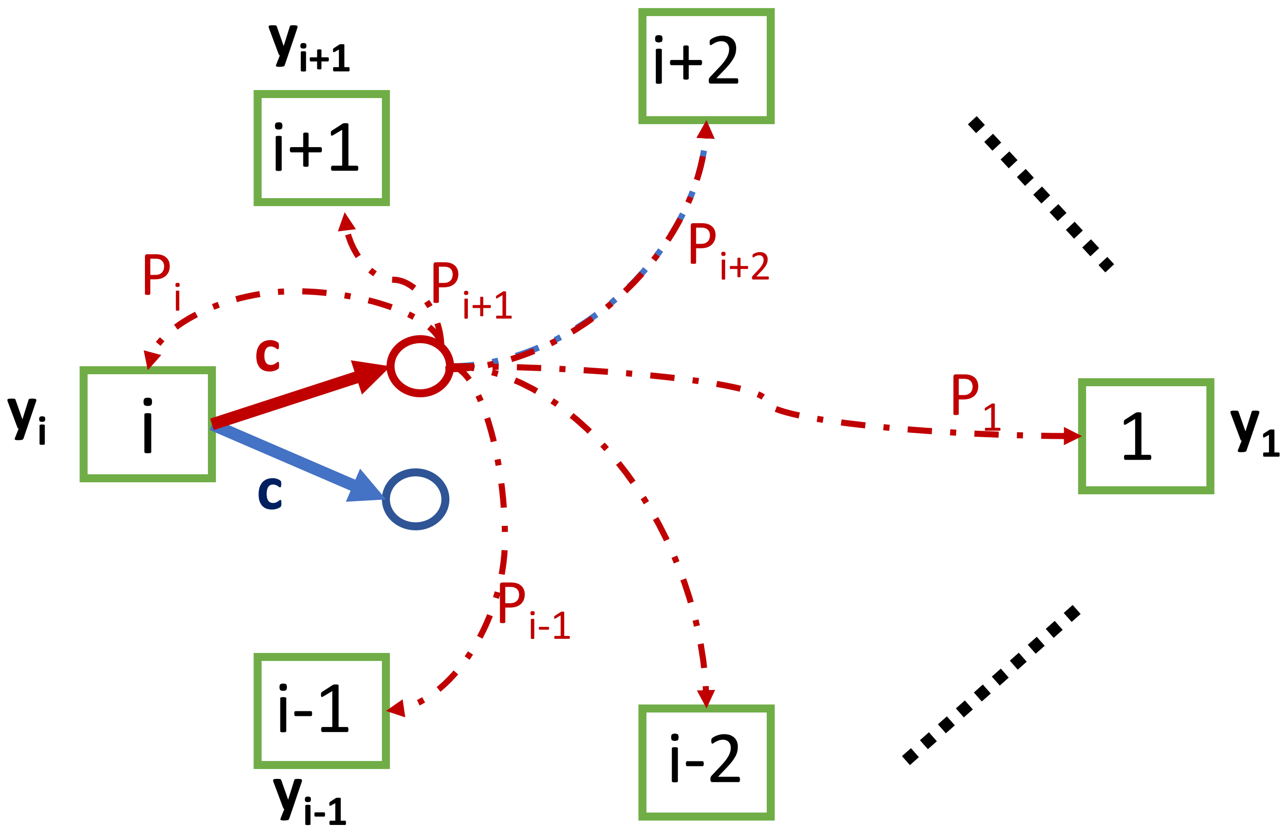
LP Formulation of the Maze Run

$$\max\ y_1 + y_2 + y_3 + y_4 + y_5 + y_6$$

s.t.
$$y_6 \le 0 + \gamma y_1$$
$$y_5 \le 1 + \gamma y_6$$
$$y_4 \le 0 + \gamma y_5$$
$$y_4 \le 0 + \gamma(0.2y_5 + y_6)$$
$$y_3 \le 0 + \gamma y_4$$
$$y_3 \le 0 + \gamma(0.5y_5 + 0.5y_6)$$
$$y_2 \le 0 + \gamma y_3$$
$$y_2 \le 0 + \gamma(0.33y_4 + 0.33y_5 + 0.33y_6)$$
$$y_1 \le 0 + \gamma y_2$$
$$y_1 \le 0 + \gamma(0.25y_3 + 0.25y_4 + 0.25y_5 + 0.25y_6)$$



MDP/RL State/Action Environment



$$c \quad + \quad \gamma \quad p^T y$$

immediate cost expect future cost

Cost-to-Go values and the LP formulation

- In general, let $y \in R^m$ represent the expected present cost-to-go values of the m states, respectively, for a given policy. Then, the cost-to-go vector of the optimal policy, with the discount factor γ, by Bellman's Principle is a Fixed Point:

$$y_i = \min\{ c_j + \gamma p_j^T y, j \in A_i \}, \forall i,$$

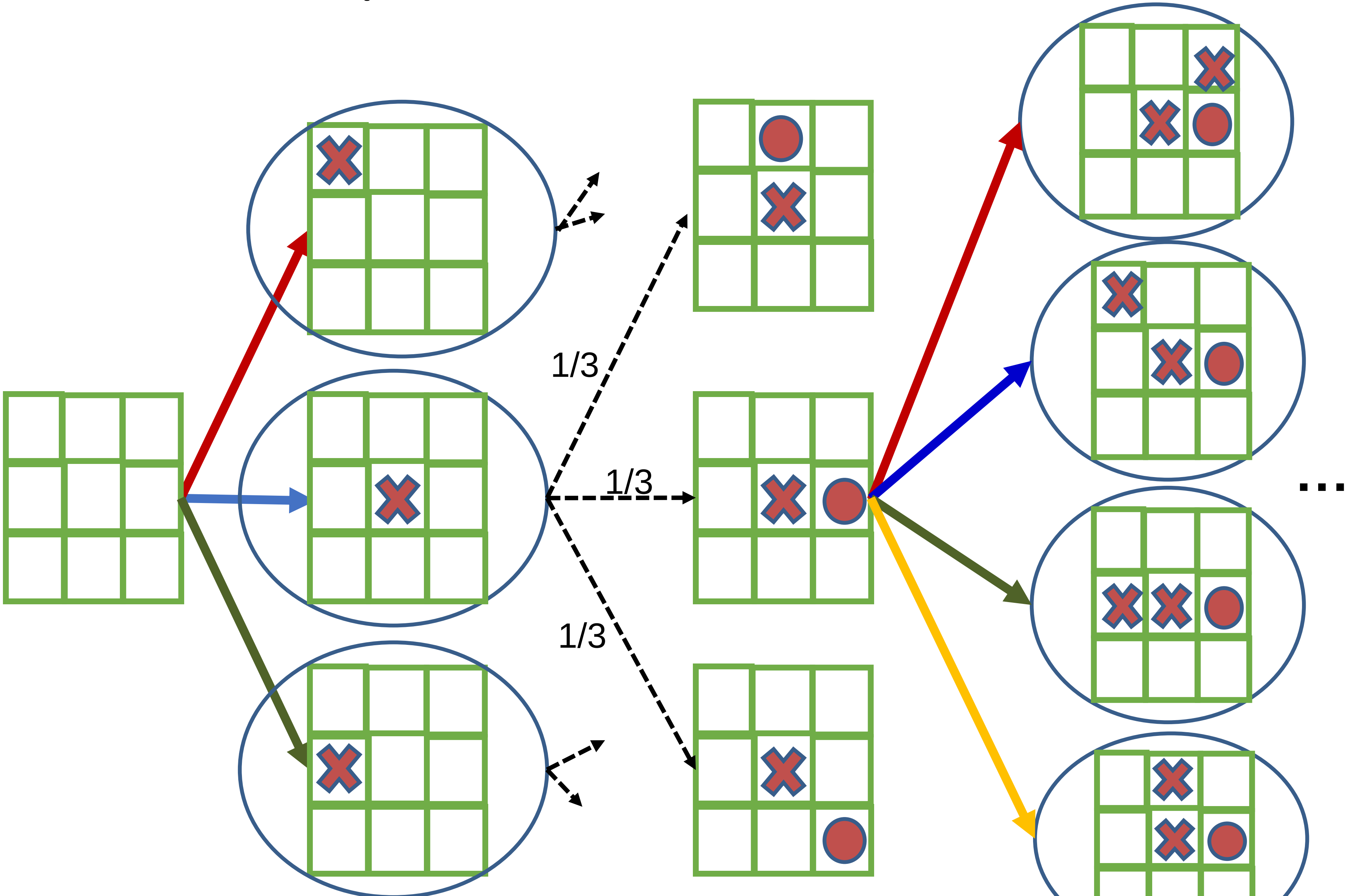
$$j_i = \arg\min\{ c_j + \gamma p_j^T y, j \in A_i \}, \forall i.$$

- Such a fixed-point computation can be formulated as an LP

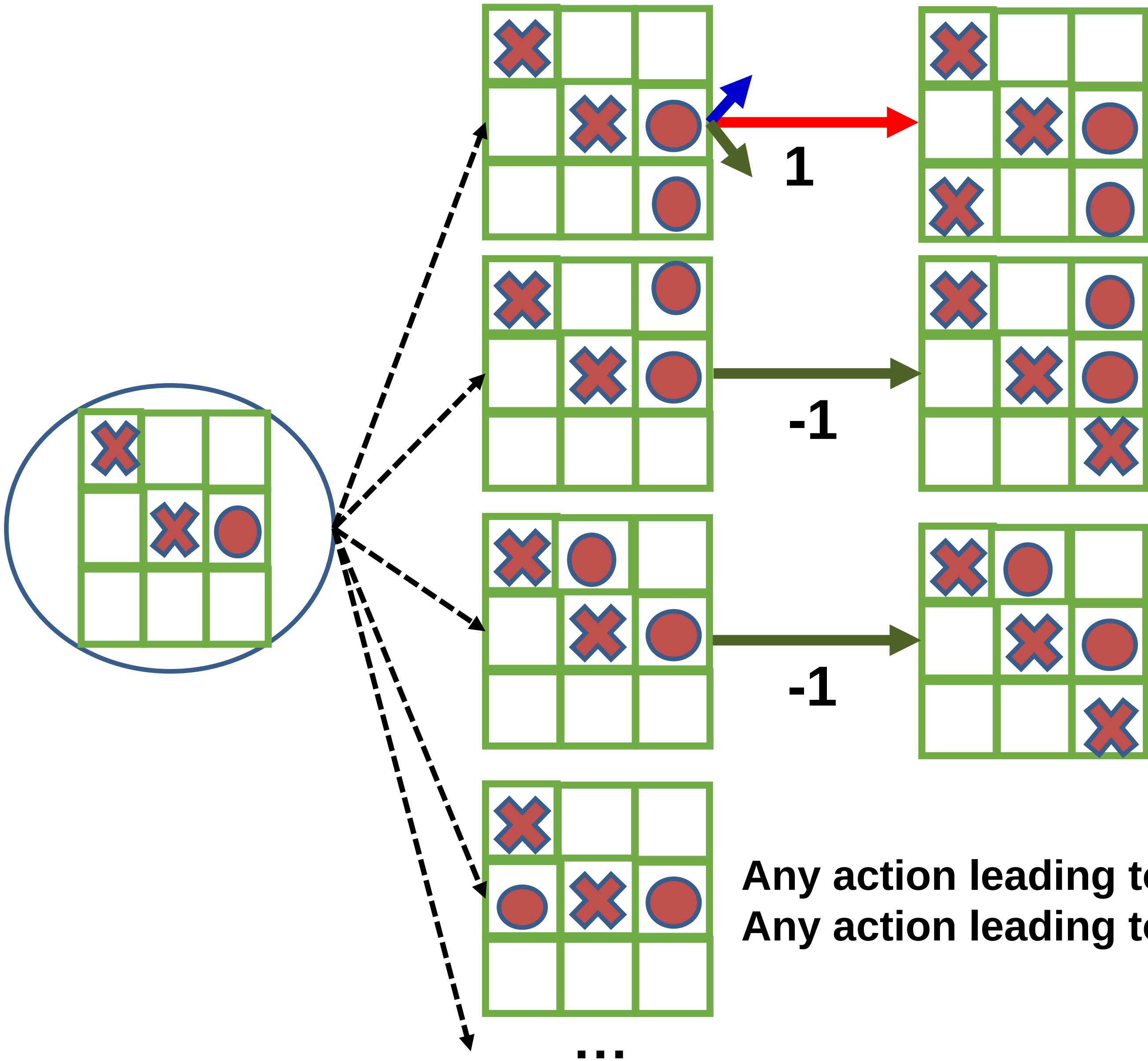
$$\begin{aligned} \max \quad & \sum_i y_i \\ \text{s.t.} \quad & y_i \le c_j + \gamma p_j^T y, \forall j \in A_i; \forall i. \end{aligned}$$

- The maximization is trying to pushing up each yi to the highest possible so that it equal to min-argument. When the optimal y is found, one can then find the index of the original optimal action/policy using argmin.

States/Actions of Tic-Tac-Toe Game



Action Costs of Tic-Tac-Toe Game



Any action leading to win has cost -1
Any action leading to lose has cost 1

LP Formulation Summary

Modeling Process

- Understand the problem
- Collect information and data
- Identify and define the decision variables
- formulate the objective function
- Isolate and formulate the constraints
- How to deal with piecewise linear and abs functions:

$|x|$ => $-z \le x \le z$, then replace $|x|$ with z in the objective

$|x|$ => $x = z' - z''$, $(z', z'') \ge 0$, then replace $|x|$ with $z' + z''$ in the objective

Business Analytics and Machine Learning/Decisions: Business Decisions based on Analytical Models and Solutions