More Linear Programming Examples I

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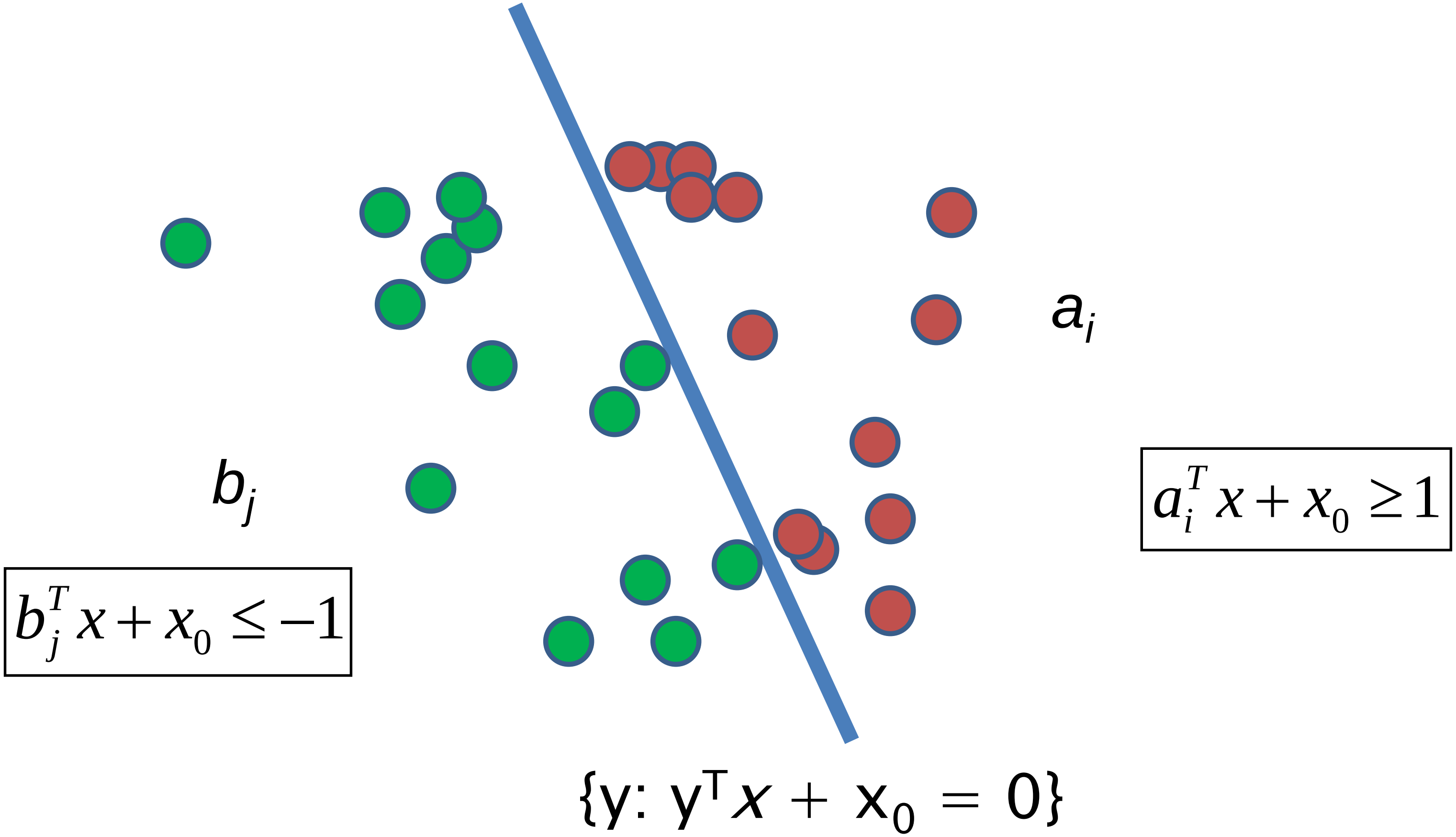
https://web.stanford.edu/class/msande211x/handout.shtml

Chapter 1.1, 1.2, 2.1, 2.2, Appendices A, B&D in Textbook

LP Example 5: Support Vector Machine

Given two sets of points (rad and blue), find a line/plane to separate them,



x is the normal direction or slope vector and x_0 is the intersect

Find a line to **strictly** separate greens and reds

LP Example 5: Is Strict Separation Possible?

$$a_i^T x + x_0 > 0, \forall i$$
$$b_j^T x + x_0 < 0, \forall j$$

Are there x and x_0 such that the following (open) inequalities are all satisfied

$$a_i^T x + x_0 \geq \varepsilon, \forall i$$
$$b_j^T x + x_0 \leq -\varepsilon, \forall j$$

Are there x and x_0 such that the following inequalities are all satisfied for arbitrarily small ε.

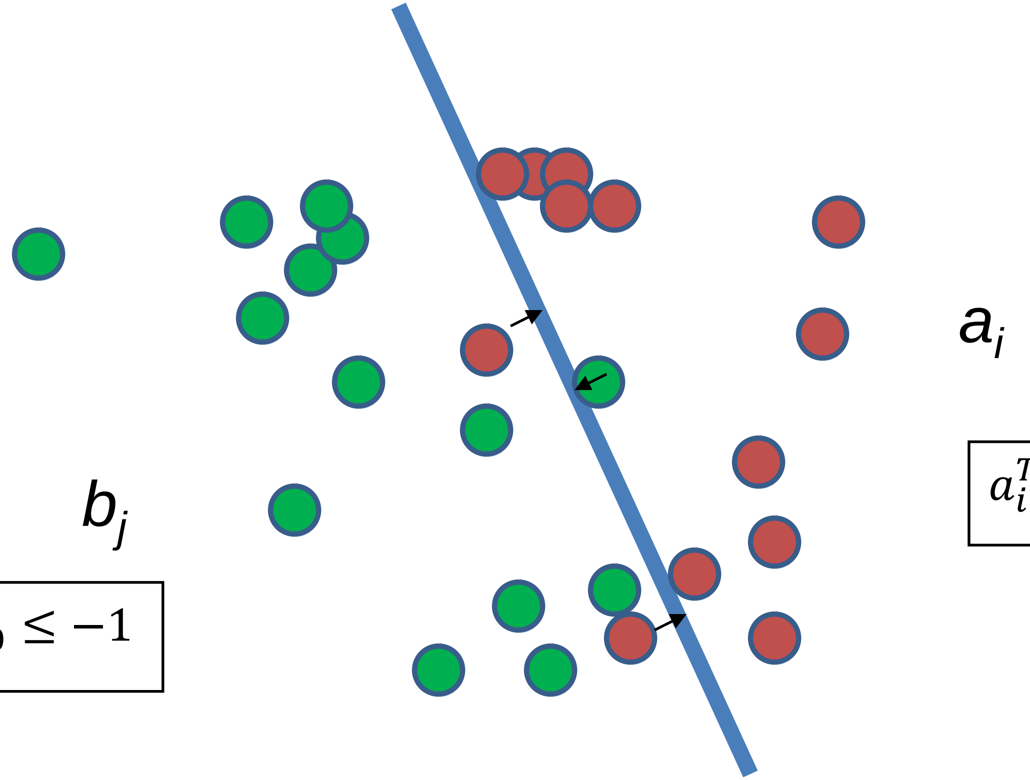
$$\Updownarrow$$

$$a_i^T x + x_0 \geq 1, \forall i$$
$$b_j^T x + x_0 \leq -1, \forall j$$

Divide x and x_0 by ε., the problem can be equivalently reformulated.

This is a special LP, called linear feasibility problem.

LP Example 5: Strict Separation is Impossible

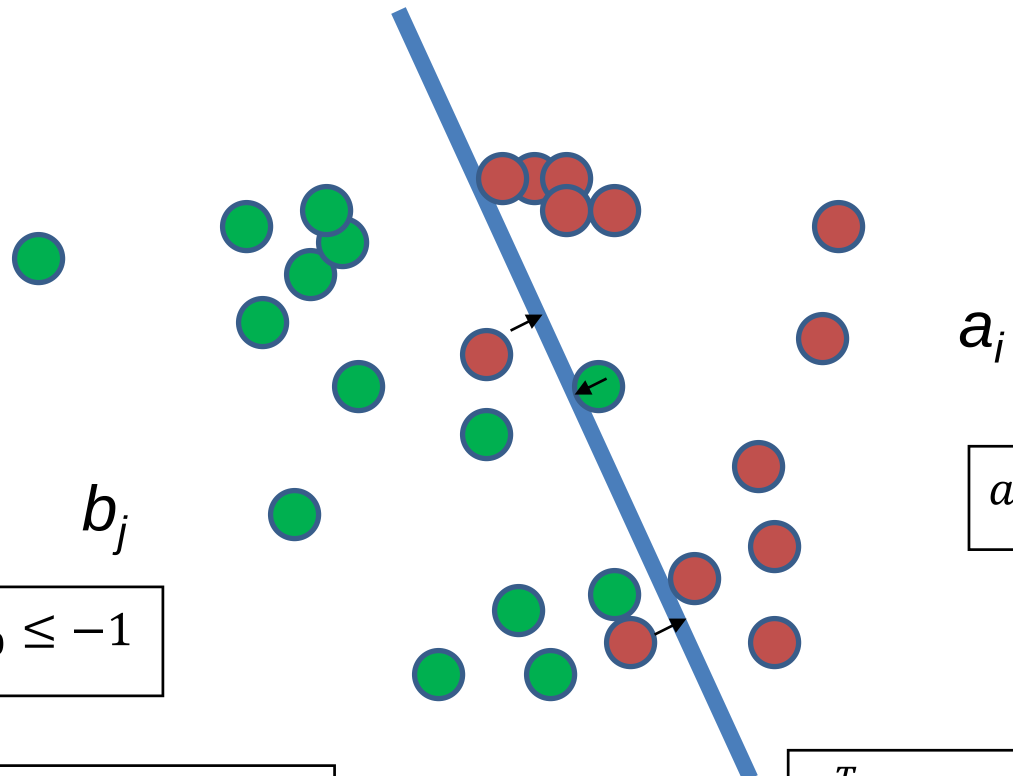


$a_i^T x + x_0 \geq 1$

$b_j^T x + x_0 \leq -1$

$$\text{minimize} \left\{ \sum_i \max\left(1 - a_i^T x - x_0\,, 0\right) + \sum_j \max(b_j^T x + x_0 + 1, 0) \right\}$$

Supporting Vector Machine Revisited



$$a_i^T x + x_0 \geq 1$$

$$a_i^T x + x_0 \geq 1 - \delta'_i, \delta'_i \geq 0$$

$$b_j^T x + x_0 \leq -1$$

$$b_j^T x + x_0 \leq -1 + \delta''_j, \delta''_j \geq 0$$

$$\text{minimize} \quad \sum_i \delta'_i + \sum_j \delta''_j$$

How to Linearize the Max Function

Introduce an auxiliary variable *w*

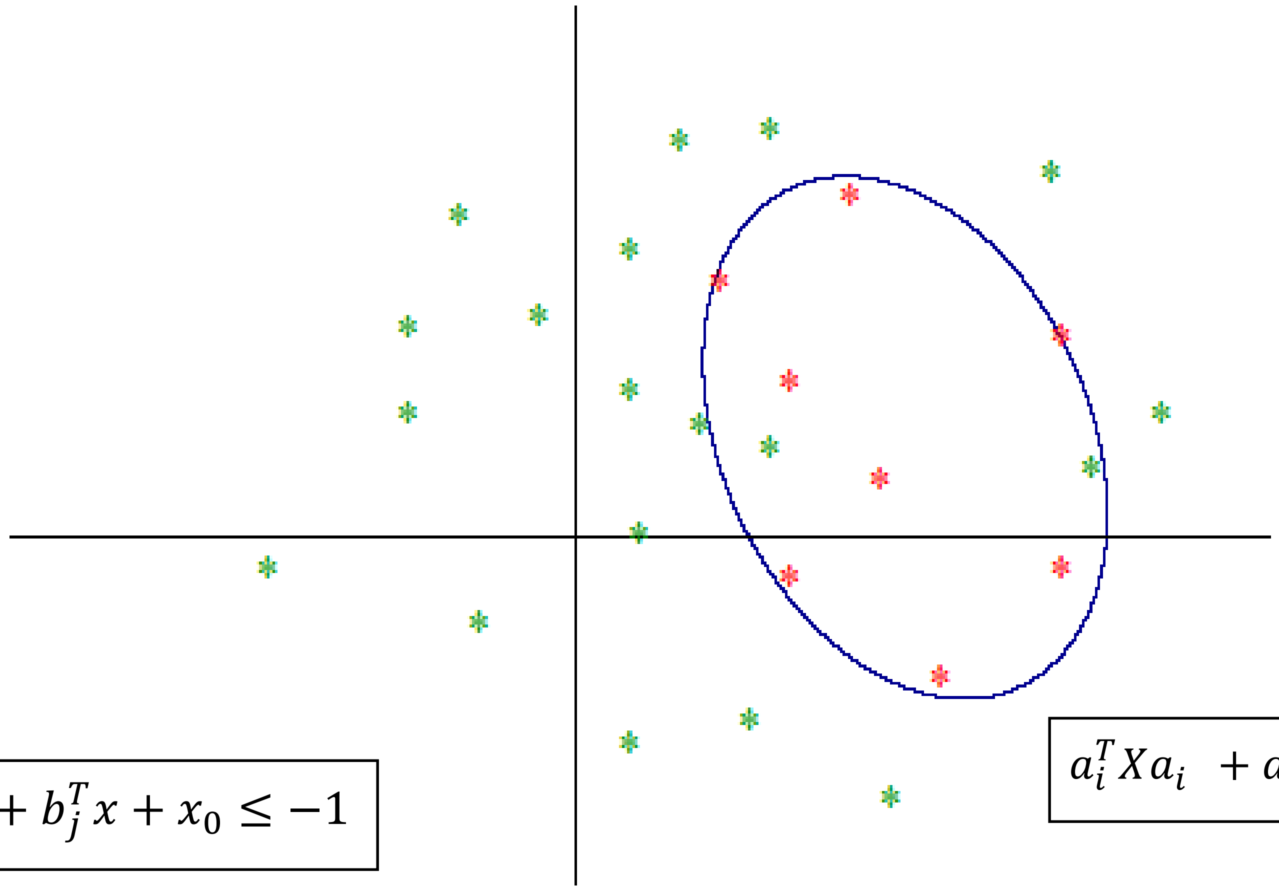
$$\max_{j=1,\ldots,m}\{\sum_i a_{ij}x_i\} = w$$

Relax it to linear inequalities

$$\sum_i a_{ij}x_i \le w, j = 1, \ldots, m$$

If *w* is minimized, the equality must hold

Supporting Vector Machine with High-Order Information



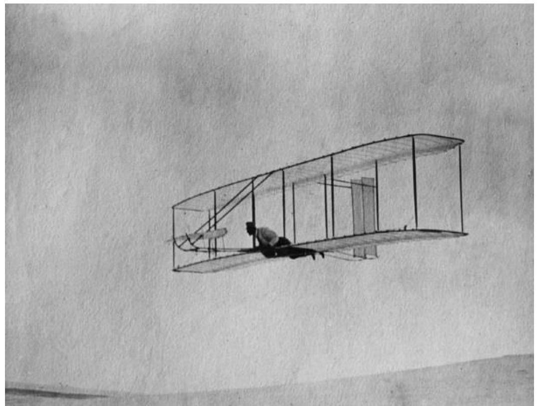
$$b_j^T X b_j \;\; + b_j^T x + x_0 \leq -1$$

$$a_i^T X a_i \;\; + a_i^T x + x_0 \geq 1$$

X is the normal direction or slope vector of Second Moment of Data-Points
This would produce a Quadratic Curve
If *X* is required to be Positive Definite, the Curve becomes an Ellipsoid.

LP Example 6: Air Traffic Control

PBS



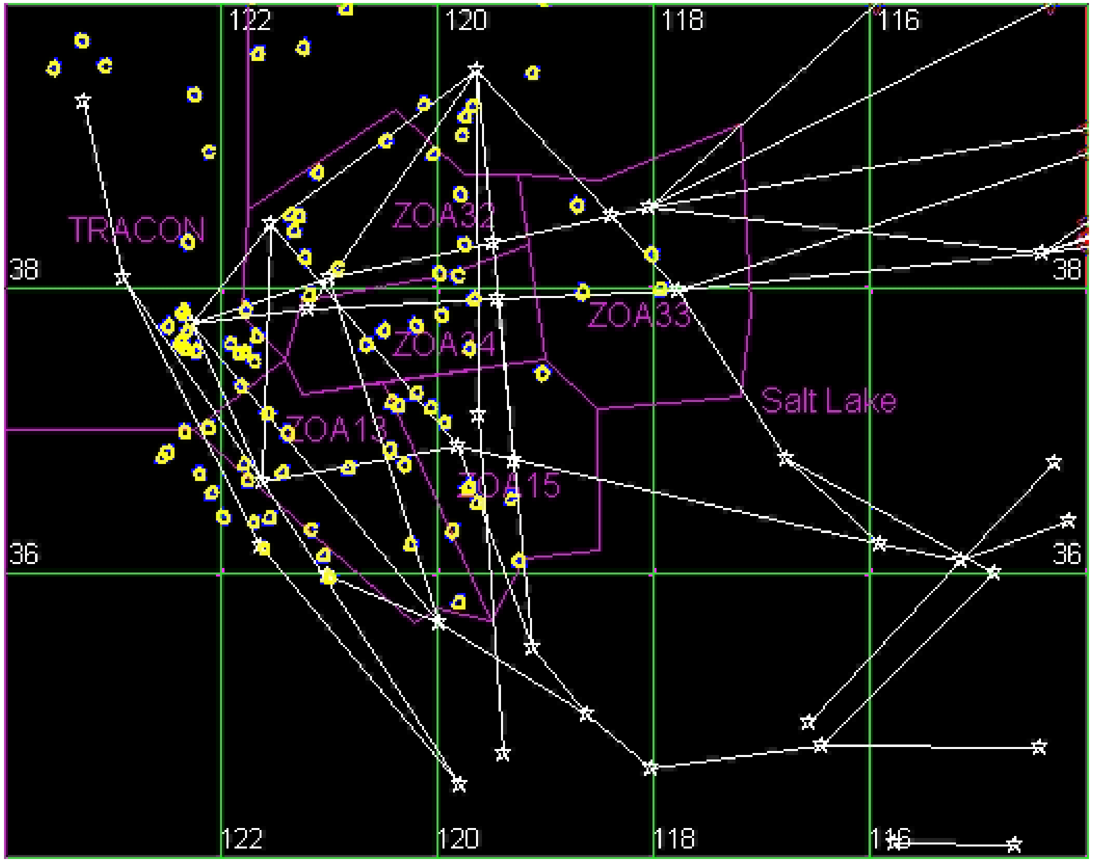
Nolan, Fundamentals of Air Traffic Control

Boeing

(AP PHOTO)

CNN

Oakland Center



Real data (playback mode)

ETMS data courtesy of NASA Ames

Air Traffic Landing Control

- Air flight j, $j = 1, \ldots, n$, must arrive at the airport within the time interval $[a_j, b_j]$ in the order of 1, 2, ..., n.
- The airport wants to find the actual arrival time for each air plane such that the narrowest metering time (inter-arrival time between two consecutive airplanes) is the greatest.
- Let: t_j be the arrival time of flight j. Then

$$\begin{array}{ll} \text{maximize} & \left[\ \min_{j=1,\ldots,n-1}\{\, t_{j+1} - t_j \}\ \right] \\ \text{s.t.} & a_j \leq t_j \leq b_j, j = 1, \ldots, n. \end{array}$$

This is not an LP problem!

How to Linearize the Min-Function

Introduce an auxiliary variable Δ

$$\min_{j=1,\ldots,n-1}\{\, t_{j+1} - t_j\} = \Delta$$

Relax it to linear inequalities

$$t_{j+1} - t_j \geq \Delta, \qquad j = 1, \ldots, n-1.$$

If Δ is maximized, the equality must hold

$$\begin{array}{ll} \max & \Delta \\ \text{s.t.} & a_j \leq t_j \leq b_j, j = 1, \ldots, n, \\ & t_{j+1} - t_j - \Delta \geq 0, j = 1, \ldots, n-1. \end{array}$$

This is an LP problem!

LP Example 7: Mechanism for Information Market

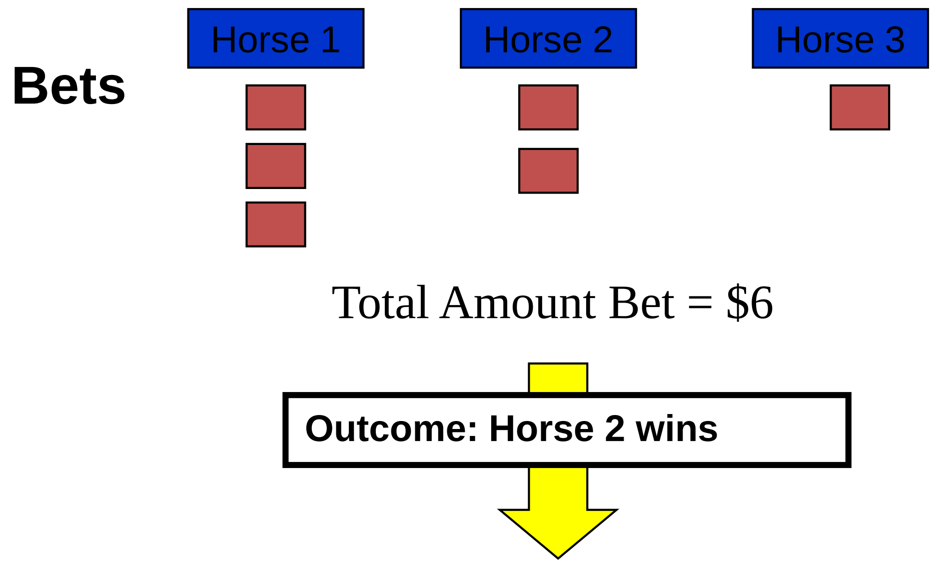
- A place where information is aggregated via market for the primary purpose of forecasting events.

- **Why:**
  - Wisdom of the Crowds: Under the right conditions groups can be remarkably intelligent and possibly smarter than the smartest person.

    James Surowiecki
  - Efficient Market Hypothesis: financial markets are "informationally efficient", prices reflect all known information
- **Market for Betting the World Cup Winner**
  - Assume 5 teams have a chance to win the World Cup: Argentina, Brazil, Italy, Germany and France

Pari-mutual Market Model: Double Auction

- Example: Pari-mutual Horseracing Betting



Winners earn $2 per bet plus stake back: Winners have stake returned then divide the winnings among themselves

Central Organization of the Market

- **Centralized Market Maker**
  - Introduce a market maker who will accept or reject orders received from participants/traders
  - Market maker may be exposed to some risk
- **Problem:** How should the market maker fill orders in such a manner that he is not exposed to any financial risk?
- **Belief-based?**
  - Central organizer will determine prices for each state based on his beliefs of their likelihood
  - This is similar to the manner in which fixed odds bookmakers operate in the betting world
  - Generally not self-funding
- **Pari-mutuel**
  - A self-funding technique popular in horseracing betting.

More Abstract Market Model

- **Market for World Cup Winner**
  - We'd like to have a standard payout of $1 per share if a participant has a winning order.
- **List of Combinatorial Orders**

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

Market maker: Order fill - how many shares to sell for each order?

More Abstract Market Model

- Given *m* states that are mutually exclusive and exactly one of them will be realized at the maturity.
- An order is a bet on one or a combination of states
  - $(a_{i1},a_{i2},\ldots,a_{im})$: the entry value is 1 if the jth state is included in the winning basket and 0 other wise.
- with a price limit
  - π_i : the maximum price the participant is willing to pay for one share of the order
- and a share quantity limit
  - q_i : the maximum number of shares the participant is willing to buy.
- A contract agreement so that on maturity it is worth a notional one dollar per share if the order includes the winning state and worth 0 otherwise.

Pari-mutual Market Model 2

Order fill	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
x1	0.75	10	1	1	1		
x2	0.35	5				1	
x3	0.40	10	1		1		1
x4	0.95	10	1	1	1	1	
x5	0.75	5		1		1	

- Let x_i be the number of shares sell to order *i*.
- The revenue collected for the sale:

$$\sum_i \pi_i x_i$$

$$0.75x_1 + \ldots + 0.75x_5$$

- The cost depends on which team wins:
  - If jth team wins (for example, if Brazil wins in the example):

$$\sum_i a_{ij} x_i$$

$$x_1 + x_4 + x_5$$

- We consider the worse case cost and profit

$$\max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\} \Longrightarrow \max \ (\sum_i \pi_i x_i - \max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\})$$

LP Pari-mutual Market Mechanism

$$\max \quad \sum_i \pi_i x_i - \max_j \{ \sum_i a_{ij} x_i \}$$
$$\text{s.t.} \quad 0 \le x_i \le q_i \qquad \forall\ i = 1, \ldots, n$$

$$\max \quad \sum_i \pi_i x_i - w$$
$$\text{s.t.} \quad \sum_i a_{ij} x_i \le w \qquad \forall\ j \in\ S$$
$$0 \le x_i \le q_i \qquad \forall\ i \in\ N$$

Collected revenue

Cost if state j is realized

Worst-case cost

This is an LP problem; later you will learn that the optimal dual solution gives prices of each team

Compact Coefficients

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

π q A

Model in Matrix Form

$$
\begin{aligned}
\max \quad & \pi^T x - w \\
\text{s.t.} \quad & A^T x - 1w \le 0, \\
& x \le q, \\
& x \ge 0
\end{aligned}
$$

1: vector of all ones

One can then use a general LP solve to solve any instance of LP problems

World Cup Betting Results and Consensus Probabilities

Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00