Optimization Models and Formulations II

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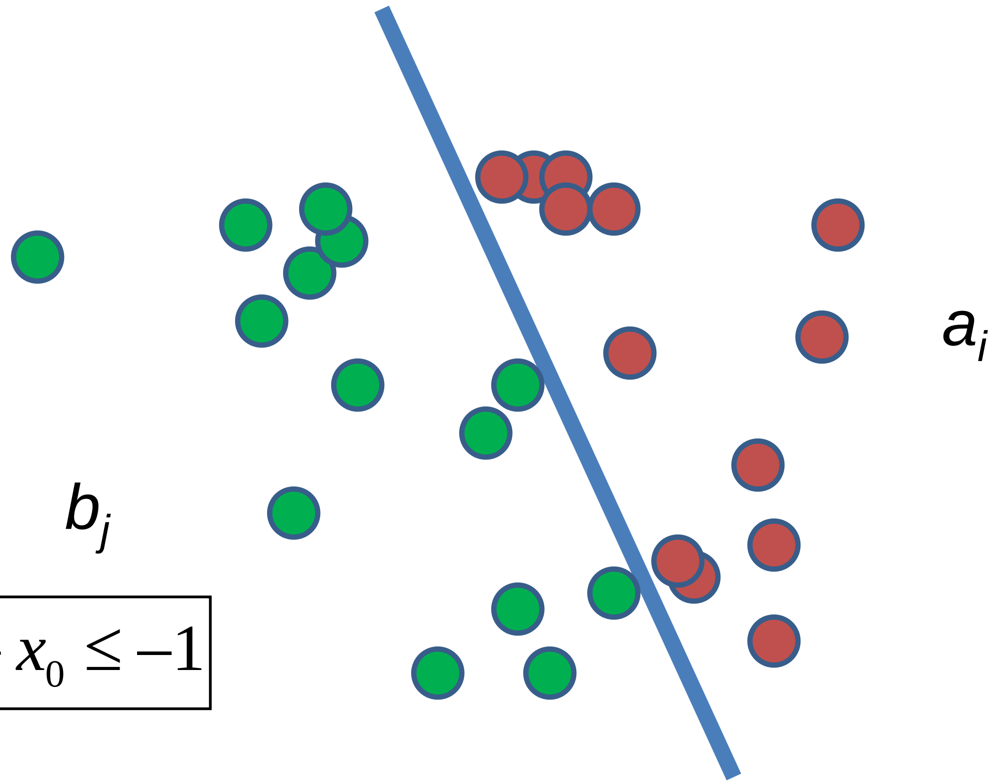
https://canvas.stanford.edu/courses/179677

Read Chapter 1.1, 1.2, 2.1, 2.2, Appendices A, B&D in Text-Book (hard copies would be available in the Book Store)

Example 5: Support Vector Machine



$$\boxed{b_j^T x + x_0 \leq -1}$$

$$\boxed{a_i^T x + x_0 \geq 1}$$

$$\{y:\ y^T x + x_0 = 0\}$$

x is the normal direction or slope vector and x_0 is the intersect

Find a line to **strictly** separate greens and reds

Is Strict Separation Possible?

$$\boxed{\begin{aligned} a_i^T x + x_0 &> 0, \forall i \\ b_j^T x + x_0 &< 0, \forall j \end{aligned}}$$

Are there x and x_0 such that the following (open) inequalities are all satisfied

$$\boxed{\begin{aligned} a_i^T x + x_0 &\geq \varepsilon, \forall i \\ b_j^T x + x_0 &\leq -\varepsilon, \forall j \end{aligned}}$$

Are there x and x_0 such that the following inequalities are all satisfied for arbitrarily small ε.

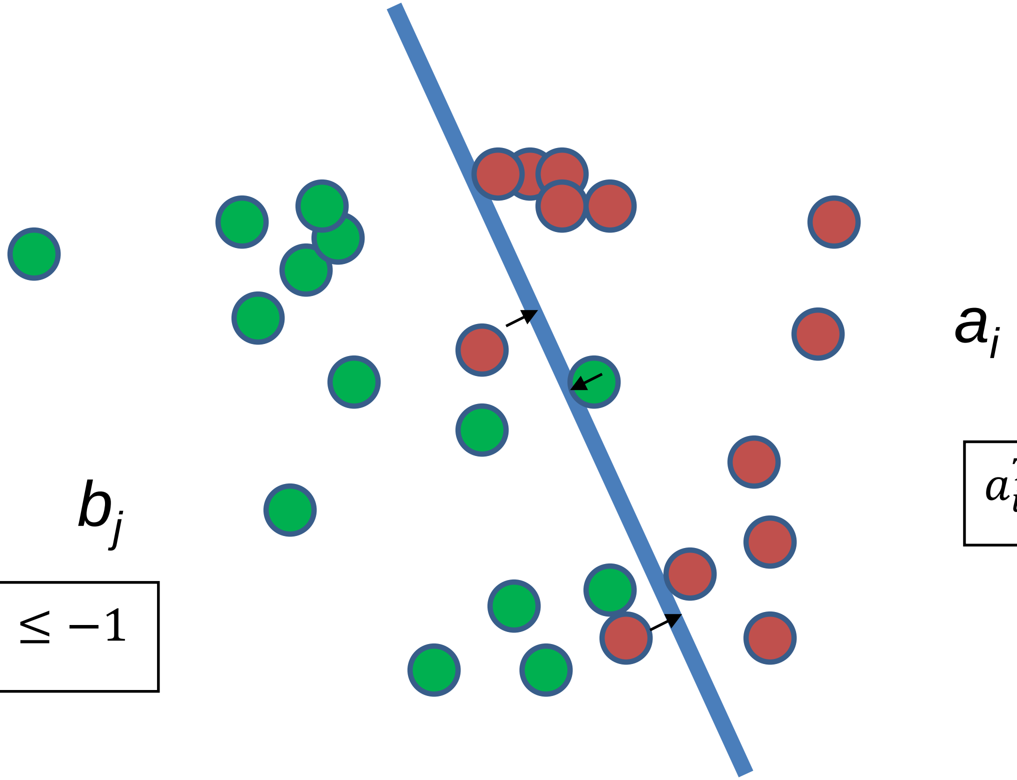
$$\Updownarrow$$

$$\boxed{\begin{aligned} a_i^T x + x_0 &\geq 1, \forall i \\ b_j^T x + x_0 &\leq -1, \forall j \end{aligned}}$$

Divide x and x_0 by ε., the problem can be equivalently reformulated.

This is a special LP, called linear feasibility problem.

When Strict Separation is Not Possible

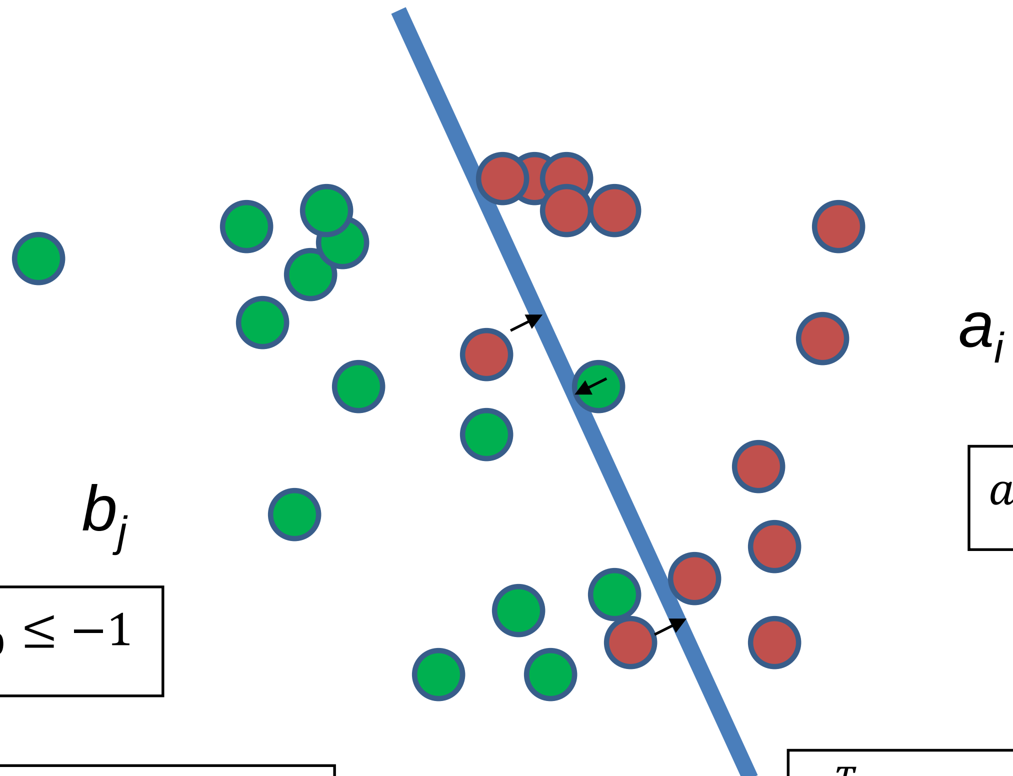


$$a_i^T x + x_0 \geq 1$$

$$b_j^T x + x_0 \leq -1$$

$$\text{minimize}\left\{ \sum_i \max\left(1 - a_i^T x - x_0\,, 0\right) + \sum_j \max(b_j^T x + x_0 + 1, 0) \right\}$$

Supporting Vector Machine Revisited



$$a_i^T x + x_0 \geq 1$$

$$a_i^T x + x_0 \geq 1 - \delta'_i, \delta'_i \geq 0$$

$$b_j^T x + x_0 \leq -1$$

$$b_j^T x + x_0 \leq -1 + \delta''_j, \delta''_j \geq 0$$

$$\text{minimize} \quad \sum_i \delta'_i + \sum_j \delta''_j$$

How to "Linearize" the Max-Function Minimization

Introduce an auxiliary variable *w*

$$\boxed{\max_{j=1,\ldots,m}\{\sum_i a_{ij}x_i\} = w}$$

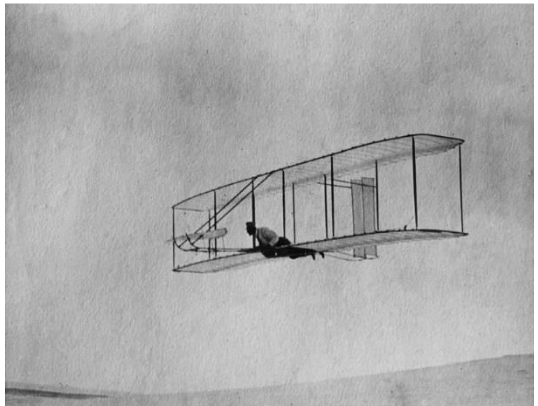
Relax it to linear inequalities

$$\boxed{\sum_i a_{ij}x_i \le w, j = 1,\ldots,m}$$

If *w* is minimized, the equality must hold

Example 6: Air Traffic Control

PBS

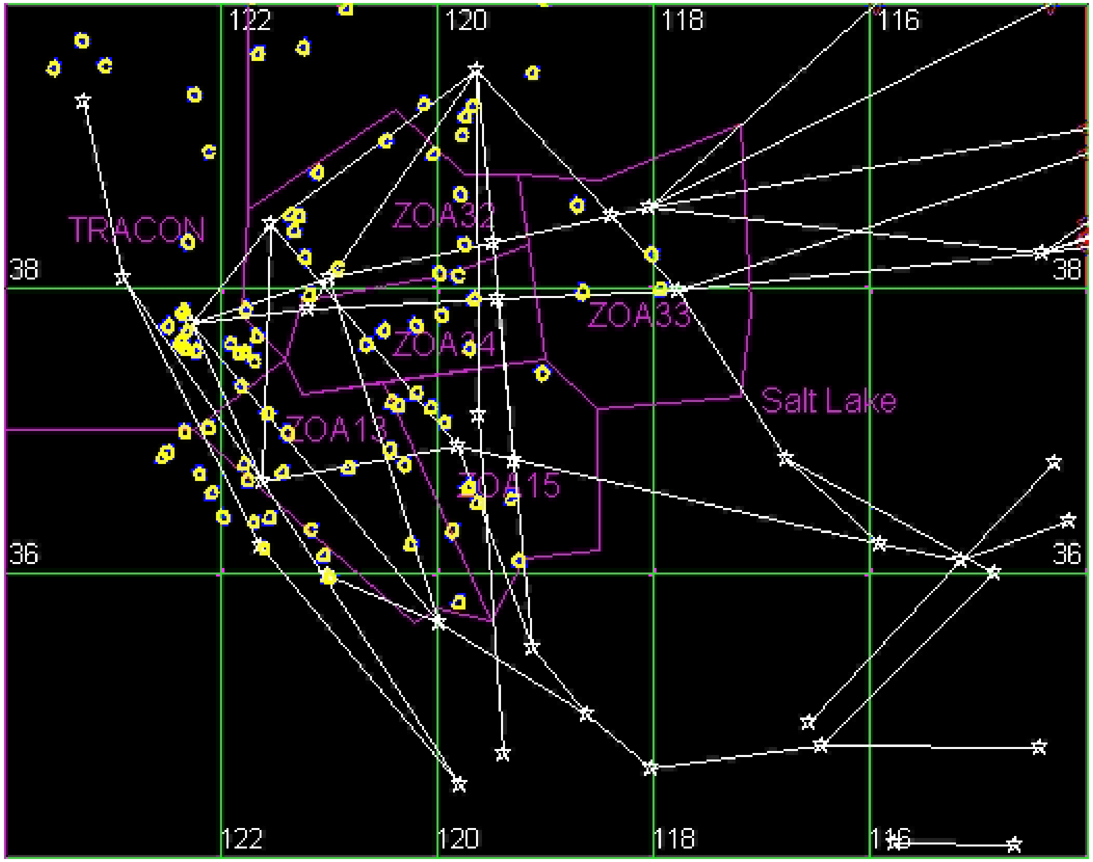


Nolan, Fundamentals of Air Traffic Control

Boeing

CNN

Oakland Center



Real data
(playback mode)

ETMS data
courtesy of NASA
Ames

Air Traffic Landing-Time Control

- Air flight j, $j = 1, \ldots, n$, must arrive at the airport within the time interval $[a_j, b_j]$ in the order of 1, 2, ..., n.
- The airport wants to find the actual arrival time for each air plane such that the narrowest metering time (inter-arrival time between two consecutive airplanes) is the greatest.
- Let: t_j be the arrival time of flight j. Then

$$\begin{array}{ll} \text{maximize} & [\ \min_{j=1,\ldots,n-1}\{\, t_{j+1} - t_j \}\ ] \\ \text{s.t.} & a_j \le t_j \le b_j, j = 1, \ldots, n. \end{array}$$

This is not an LP problem!

How to “Linearize” the Min-Function Maximization

Introduce an auxiliary variable Δ

$$\boxed{\min_{j=1,\ldots,n-1}\{\, t_{j+1} - t_j\} = \Delta}$$

Relax it to linear inequalities

$$\boxed{t_{j+1} - t_j \geq \Delta, \qquad j = 1, \ldots, n-1.}$$

If Δ is maximized, the equality must hold

$$\begin{array}{ll} \max & \Delta \\ \text{s.t.} & a_j \leq t_j \leq b_j, j = 1, \ldots, n, \\ & t_{j+1} - t_j - \Delta \geq 0, j = 1, \ldots, n-1. \end{array}$$

This is an LP problem!

Example 7: Unsupervised Learning

Build a model that will predict a probability for each credit card transaction indicating whether the transaction is business or personal related.

- There is no training data where particular transactions are identified as being personal, we used personal remittances as the best proxy
- On the transaction side, we focused on the industry code of each transaction as a key initial differentiator between transactions
- Developed a LP model to establish probabilities for each industry code that indicate the likelihood that dollars spent in that code will be personal spending.

Transaction Types with Industrial Codes

Industry Code	Description
995	CLUB - WAREHOUSE
25	DEPARTMENT STORE - MASS MERCHANDISER
728	GASOLINE/OIL COMPANY - NATIONAL DEALER
729	GASOLINE/OIL COMPANY - INDEPENDENT DEALER
429	SHOP - HOME IMPROVEMENT
415	DEPARTMENT STORE - FULL SERVICE
87	INTERNET TRAVEL
504	SHOP - ELECTRONIC GOODS
616	COMMUNICATION - CABLE & BROADCAST SERVICES
215	AUTO SERVICES - MOTOR RELATED SERVICES/DEALER
404	AUTO SERVICES - AUTO SALES & SERVICE
443	SHOP - SPORTING GOODS
457	SHOP - CHEMIST/PHARMACY
522	SHOP - FURNITURE
463	SHOP - JEWELRY
757	ENTERTAINMENT - TICKET AGENT - COMPANY
407	SHOP - CLOTHING - FAMILY
680	SHOP - COMPUTER HARDWARE
465	SHOP - LIQUOR STORE
400	AUTO SERVICES - VEHICLE ACCESSORIES
416	DEPARTMENT STORE - SPECIALITY
428	SHOP - HOME FURNISHINGS
414	SHOP - CLOTHING - WOMEN'S
793	TRAVEL - TOUR OPERATOR GENERAL
412	SHOP - CLOTHING - MEN'S & WOMEN'S
787	TRAVEL - NON - `AGENT RETAILER
447	SHOP - SHOES - MEN'S ONLY
427	SHOP - HARDWARE/DO IT YOURSELF
554	MAIL ORDER SELF IMPROVEMENT/BUSINESS SEMINARS
603	SERVICES - BEAUTY SHOPS/BEAUTICIAN

Data Analytics: Business or Personal?

For each of the industry codes, the model will determine a probability which indicates the likelihood that a transaction was personal.

Each Column represents an Industry Code

Personal Remittances

				…					
Account	**1**	**2**	**3**	…	**n**			**Actual**	
1	**$156**	**$0**	**$87**		**$25**			**$200**	
2	**$200**	**$25**	**$0**		**$0**			**$195**	
…	**$0**	**$134**	**$35**		**$60**			**$210**	

Value of transactions in period

Probability Estimation as Decision Variables

For each of the industry codes, the model will determine a probability (in red) which indicates the likelihood that a transaction was personal. The goal is to minimize the sum of the squares of the differences (in blue).

Probability Personal

Each Column represents an Industry Code

Personal Remittances

	25%	**10%**	**0%**	**…**	**5%**				
Account	**1**	**2**	**3**	**…**	**n**		**Predicted**	**Actual**	**Difference**
1	**$156**	**$0**	**$87**		**$25**		**$244**	**$200**	**$44**
2	**$200**	**$25**	**$0**		**$0**		**$200**	**$195**	**$5**
…	**$0**	**$134**	**$35**		**$60**		**$230**	**$210**	**$20**

Value of transactions in period

Regression Optimization Model

Our model will determine the probability that a transaction from each industry code is personal in such a manner which will minimize the sum of the squared errors (between predicted personal remittances and actual personal remittances).

$$\text{Min} \quad \sum_i \left| \sum_j a_{ij} x_j - b_i \right|^p$$

$$\text{s.t.} \quad 0 \leq x_j \leq 1, \forall j.$$

- Let x_j be such a probability that a transaction is personal for industry code j
- $a_{i,j}$ – transaction amount for account i and industry code j
- b_i – amount paid by personal remit for account i
- $\sum_j a_{i,j} x_j$ – the expected personal expenses for account i
- We'd like to choose x_j such that $\sum_j a_{i,j} x_j$ matches b_i for ALL i, where p=1 or 2
- P=1 leads to LP

How to "Linearize" the Abs-Function Minimization

To dealing the abs function, we introduce auxiliary variables y_i

$$\boxed{|z_i| = y_i, i = 1, \ldots, m.}$$

Relax it to linear inequalities

$$\boxed{-y_i \leq z_i \leq y_i, i = 1, \ldots, m.}$$

If the sum of y_is is minimized, the equality must hold

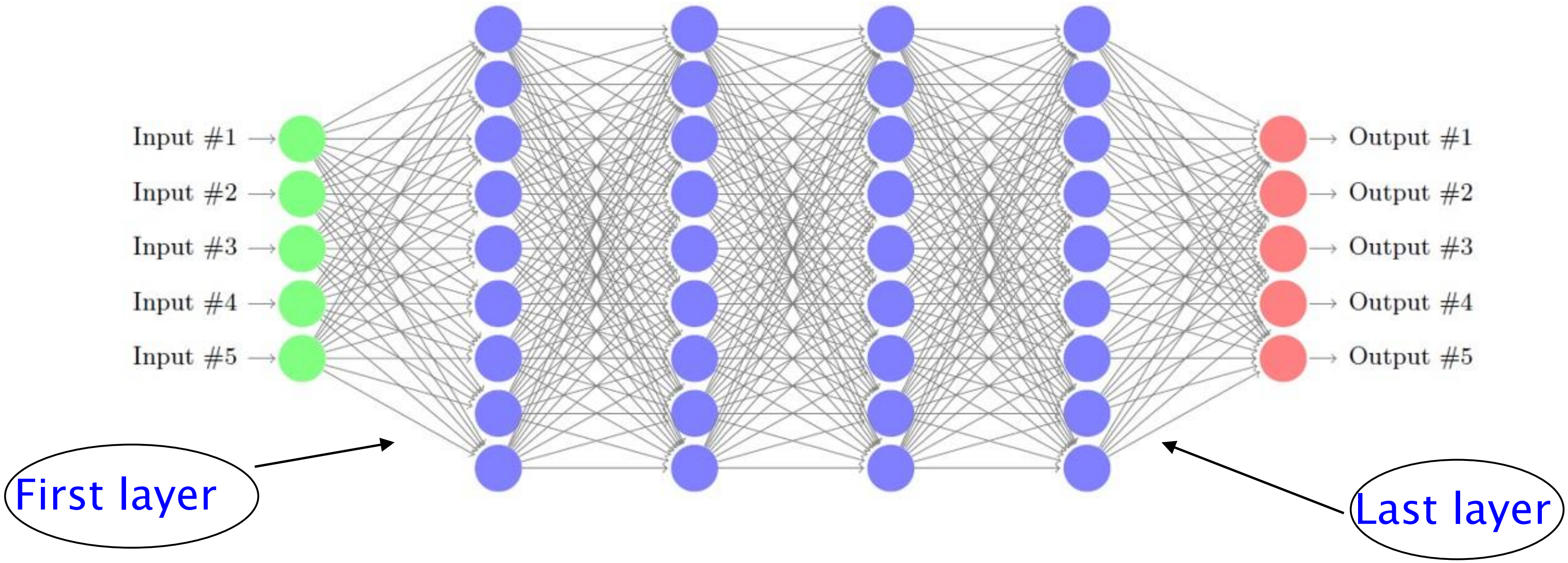
$$
\begin{array}{ll}
\min & \displaystyle\sum_{i=1}^{m} y_i \\
\text{s.t.} & -y_i \leq \displaystyle\sum_{j} a_{ij} x_j - b_i \leq y_i, \forall i \\
& 0 \leq x_j \leq 1, \forall j.
\end{array}
$$

This is an LP problem!

$$
\begin{array}{c}
-y \leq Ax - b \leq y, \\
0 \leq x \leq 1
\end{array}
$$

Nonlinear Regression: Bird or Plane?

Neural Network Design for Supervised Prediction



$$\text{optimize} \quad F(w_{i,j})$$

where $w_{i,j}$ is the weigh variable at laye i and edge j,

from a training set of pairs of inputs - outputs data so that when a new input data come the system predict what output would be.

Example 8: Information Market

- A place where information is aggregated via market for the primary purpose of forecasting events.

- **Why:**
  - Wisdom of the Crowds: Under the right conditions groups can be remarkably intelligent and possibly smarter than the smartest person.

    James Surowiecki
  - Efficient Market Hypothesis: financial markets are "informationally efficient", prices reflect all known information
- **Market for Betting the World Cup Winner**
  - Assume 5 teams have a chance to win the World Cup: Argentina, Brazil, Italy, Germany and France

Optimizations for the Market

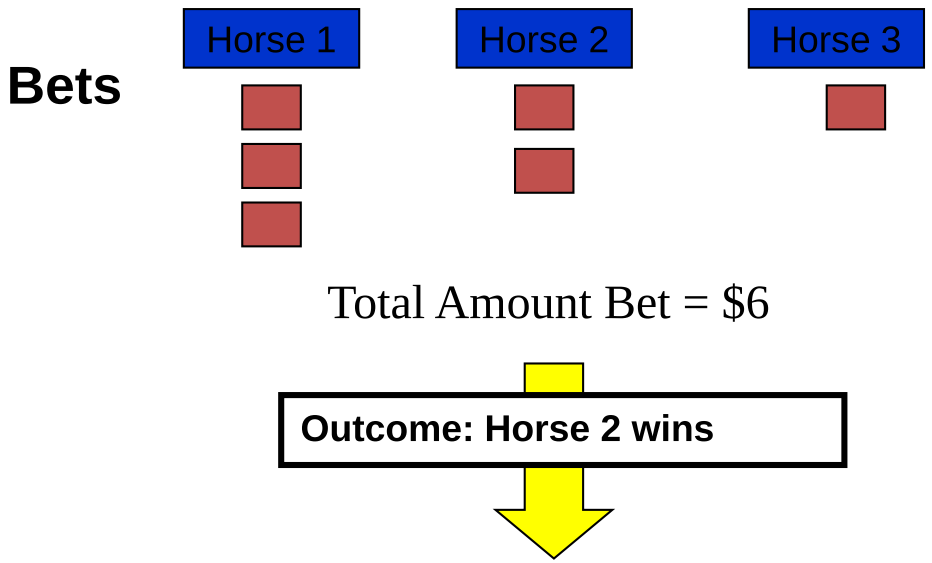
- **Double Auction:** Let participants trade directly with one another
  - Requires participants to find someone to take the other side of their order (i.e.: the complement of the set of teams which they have selected)
- **Centralized Market Maker**
  - Introduce a market maker who will accept or reject orders received from participants/traders
  - Market maker may be exposed to some risk
- **Problem:** How should the market maker fill orders in such a manner that he is not exposed to any financial risk?

Central Organization of the Market

- **Belief-based**
  - Central organizer will determine prices for each state based on his beliefs of their likelihood
  - This is similar to the manner in which fixed odds bookmakers operate in the betting world
  - Generally not self-funding
- **Pari-mutuel**
  - A self-funding technique popular in horseracing betting.

Pari-mutual Market Model 1

- Example: Pari-mutual Horseracing Betting



Winners earn $2 per bet plus stake back: Winners have stake returned then divide the winnings among themselves

More Abstract Market Model

- **Market for World Cup Winner**
  - We'd like to have a standard payout of $1 per share if a participant has a winning order.
- **List of Combinatorial Orders**

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

Market maker: Order fill - how many shares to sell for each order?

More Abstract Market Model

- Given m states that are mutually exclusive and exactly one of them will be realized at the maturity.
- An order is a bet on one or a combination of states
  - $(a_{i1}, a_{i2}, \ldots, a_{im})$: the entry value is 1 if the jth state is included in the winning basket and 0 other wise.
- with a price limit
  - π_i : the maximum price the participant is willing to pay for one share of the order
- and a share quantity limit
  - q_i : the maximum number of shares the participant is willing to buy.
- A contract agreement so that on maturity it is worth a notional one dollar per share if the order includes the winning state and worth 0 otherwise.

Pari-mutual Market Model 2

- Let x_i be the number of shares sell to order i.
- The revenue collected for the sale:

$$\sum_i \pi_i x_i \qquad 0.75x_1 + \ldots + 0.75x_5$$

Order fill	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
x1	0.75	10	1	1	1		
x2	0.35	5				1	
x3	0.40	10	1		1		1
x4	0.95	10	1	1	1	1	
x5	0.75	5		1		1	

- The cost depends on which team wins:
  - If jth team wins (for example, if Brazil wins in the example):

$$\sum_i a_{ij} x_i \qquad x_1 + x_4 + x_5$$

- We consider the worse case cost and profit

$$\max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\} \Longrightarrow \max \ (\sum_i \pi_i x_i - \max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\})$$

LP Pari-mutual Market Mechanism

$$\max \quad \sum_i \pi_i x_i - \max_j \{ \sum_i a_{ij} x_i \}$$
$$\text{s.t.} \quad 0 \le x_i \le q_i \qquad \forall\ i = 1, \ldots, n$$

$$\max \quad \sum_i \pi_i x_i - w$$
$$\text{s.t.} \quad \sum_i a_{ij} x_i \le w \qquad \forall\ j \in\ S$$
$$0 \le x_i \le q_i \qquad \forall\ i \in\ N$$

Collected revenue

Worst-case cost

Cost if state j is realized

This is an LP problem; later you will learn that
the optimal dual solution gives prices of each team

World Cup Betting Results

Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

By-Product Outcome: State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00