VinU Mini-Lecture
Introduction to Optimization
Homework 7
Course Instructor: Yinyu Ye

Problem 1

Consider a variant of the Two-Person Zero-Sum Matrix Game in Slides 2-4 of Lecture Note #8, where the payoff matrix becomes:

$$P = \begin{bmatrix} 4 & -1 & -4 & -2 \\ -2 & 1 & 4 & 2 \\ 1 & -2 & 2 & -4 \end{bmatrix}$$

a) Write down the linear program for Player Row.

b) Write down the dual of the above linear program.

c) Give interpretations of the dual problem (with respect to the meaning of the dual variables and dual objective).

Problem 2

True or False: in linear programming, strong duality always holds if both of the primal and the dual problems are feasible.

Problem 3

Consider the linear program (P) of the form

$$\begin{array}{ll} \text{minimize} & \mathbf{q}^T\mathbf{z} \\ \text{subject to} & \mathbf{M}\mathbf{z} \geq -\mathbf{q} \\ & \mathbf{z} \geq \mathbf{0} \end{array}$$

in which the matrix $\mathbf{M}$ is *skew symmetric*; that is, $\mathbf{M} = -\mathbf{M}^T$.

1. Show that problem (P) and its dual are the same.

2. A problem of the kind in part (a) is said to be *self-dual*. An example of a self-dual problem has

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}.$$

   Give an interpretation of the problem with this data.

3. Explain that a self-dual linear program has an optimal solution if and only if it is feasible.

Problem 4 Bonus Problem

Consider a finite set I of firms that are making production decisions regarding n products in order to maximize profits subject to linear resource constraints, where each firm solves an LP

$$\begin{aligned} \max_{\mathbf{x}_i} \quad & \mathbf{c}^\top \mathbf{x}_i \\ \text{s.t.} \quad & A\mathbf{x}_i \leq \mathbf{b}^i \\ & \mathbf{x}_i \geq 0. \end{aligned}$$

Here $\mathbf{c} \in R^n_+$ is the unit-profit vector corresponding to the n products, which are the same across all firms, $\mathbf{b}^i \in R^m_+$ is the amount of m resources available to firm i, and A is the common production matrix where $A_{j,k}$ encodes the amount of resource j required in the production of one unit of product k.

Now suppose that firms are allowed to make alliances among themselves and pool resources with other firms in the same alliance. The collective production decision problem that an alliance $S \subset I$ faces is then given by

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \leq \sum_{i \in S} \mathbf{b}^i \\ & \mathbf{x} \geq 0 \end{aligned}$$

Let V^S be the resulting maximum profit (the optimal objective) for alliance S's LP. We refer to $S = I$ as the **Grand Alliance**. The **core** of the Grand Alliance is the set of payment vectors $\mathbf{z} = (z_1, \ldots, z_{|I|})$ (where z_i is the profit payment to firm i) that satisfy

$$\begin{aligned} & \sum_{i \in I} z_i = V^I \\ & \sum_{i \in S} z_i \geq V^S, \forall S \subset I, \end{aligned}$$

i.e. the Grand Alliance profit is divided among the $|I|$ firms, and the collective profit for any subset of firms weakly dominates the optimal profit achieved by that subset if they formed an alliance for production by themselves. Core is a fundamental concept in Collaborative Game Theory.

1. Write the dual problem of the LP faced by alliance $S \subset I$ and the optimality conditions.

2. Provide an interpretation of the optimal dual variables of the Grand Alliance LP.

3. Let $\mathbf{y}^*$ be an optimal dual solution of the Grand Alliance LP problem, and consider the profit allocation $z_i^* = {\mathbf{y}^*}^\top \mathbf{b}^i$ for all i. Show that $\mathbf{z}^* = (z_1^*, \ldots, z_{|I|}^*)$ is in the core, that is, $\mathbf{z}^*$ satisifes criteria of the core (Hint: use the strong and weak duality theorems).