MS&E 111X & 211X
Introduction to Optimization (Accelerated)
Homework 3
Course Instructor: Yinyu Ye
Due Date: 11:59 pm Nov 4, 2021

Please submit your homework through Gradescope. If you haven't already been added to Gradescope, you can use the entry code **2RJNKV** to join. Please note: late homework will not be accepted. Each problem will be graded out of 10 points.
Some problems allow group work. Groups should be no larger than 4. If you decide to work together, provide the names of those you worked with.

Problem 1

The Cobb-Douglas production function is widely used in economics to represent the relationship between inputs and outputs of a firm. It takes on the form $Y = AL^{\alpha}K^{\beta}$ where Y represents output, L labor and K capital. α and β are constants that determine how production is scaled. We find that the Cobb-Douglas function can be applied to model firm utility. Consider the following utility maximization problem:

$$\begin{aligned} \max \quad & u(x) = x_1^{\alpha} x_2^{1-\alpha} \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 \leq w \\ & x_1, x_2 \geq 0 \end{aligned}$$

where $0 < \alpha < 1$ is fixed and p and w represent a given price and budget respectively. This is a particular instance of Cobb-Douglas utility. Assume that $w > 0$ and $p > 0$.

a) Perform a logarithmic transformation to find an equivalent maximization problem. Explain why this transformation leads to an equivalent problem.

b) Write the KKT conditions for the transformed problem and find an explicit solution for x as a function of p, w and α. Are these conditions sufficient for optimality?

c) Find the corresponding Lagrangian multiplier, λ, in terms of p, w and α. Describe an interpretation for λ.

d) Suppose $w = 100$, $p_1 = 1$, $p_2 = 2$, and $\alpha = 0.2$, find the optimal consumption bundle x_1 and x_2.

Problem 2

Consider the optimization problem

$$\begin{aligned} \min \quad & x_1^2 + 4x_2^2 \\ \text{subject to} \quad & x_1^2 + 2x_2^2 \geq 4 \end{aligned}$$

a) Find all points that satisfy the KKT conditions.

b) Apply the second order condition to determine the whether or not the KKT solutions are local minimizers or maximizers or neither.

Problem 3

Consider the maze run MDP problem in Problem 4 of Homework 1, also shown here in Figure 1. In Homework 1, we formulated this problem as a linear program where the decision variables are the cost-to-go-values of the decision states.

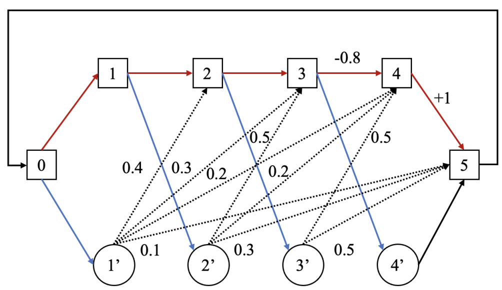

Figure 1: Modified Maze Run

a) Write down the dual problem of this linear program, solve it using your favorite solver (in teams), and give some interpretations about these dual variables.

b) Give an optimal policy for this Reinforcement Learning problem (Hint: use the dual optimal solution and the complementarity conditions).

Problem 4

Consider a variant of the Two-Person Zero-Sum Matrix Game in Slides 2-4 of Lecture Note #8, where the payoff matrix becomes:

$$P = \begin{bmatrix} 4 & -1 & -4 & -2 \\ -2 & 1 & 4 & 2 \\ 1 & -2 & 2 & -4 \end{bmatrix}$$

a) Write down the linear program for Player Row.

b) Write down the dual of the above linear program.

c) Give interpretations of the dual problem (with respect to the meaning of the dual variables and dual objective).

Problem 5

Consider a variant of the Robust Portfolio Management Problem in Slides 19-23 of Lecture Note #8, where constraints on x_1, x_2 become:

$$x_1 + x_2 = 1$$

$$3x_1 - x_2 \geq 0$$

and constraints on μ_1, μ_2 become

$$\mu_1 + 3\mu_2 = 2$$

$$|\mu_1 - \mu_2| \leq 1$$

a) Write its inner problem as a linear program.

b) For fixed x_1 and x_2 (under the constraints of $x_1 + x_2 = 1, 3x_1 - x_2 \geq 0$), find the dual of the inner problem, and solve dual problem.

c) Combine the objectives of the outer and inner problem into a joint single layer problem.

Problem 6

For parts a)-c) below, label them as True or False. If true, provide a short reason; if false, provide reasoning or a counter example.

a) The shadow price of a non-binding constraint can be non-zero.

b) For a LP problem, it is possible that the primal problem has an unbounded objective value, while the dual problem has a non-empty feasible region.

c) For a LP problem, it is possible that the primal problem has a finite optimal value, while the dual problem has no feasible solution.