MS&E 111X & 211X
Introduction to Optimization (Accelerated)
Homework 1
Course Instructor: Yinyu Ye
Due Date: 5:00 pm Oct 7, 2021

Please submit your homework through Gradescope. If you haven't already been added to Gradescope, you can use the entry code **2RJNKV** to join. Please note: late homework will not be accepted. Each problem will be graded out of 10 points.
Some problems allow group work. Groups should be no larger than 4. If you decide to work together, provide the names of those you worked with.

Problem 1

(Text page 28 problem 3) An oil refinery has two sources of crude oil: a light crude that costs $ 35/barrel and a heavy crude that costs $ 30/barrel. The refinery produces gasoline, heating oil, and jet fuel from crude in the amounts per barrel indicated in the following table:

	Gasoline	Heating Oil	Jet Fuel
Light Crude	0.3	0.2	0.3
Heavy Crude	0.3	0.4	0.2

The refinery has contracted to supply 900,000 barrels of gasoline, 800,000 barrels of heating oil, and 500,000 barrels of jet fuel. The refinery wishes to find the amounts of light and heavy crude to purchase so as to be able to meet its obligations at minimum cost.

1. Formulate this problem as a linear program.
2. Solve the LP with a solver, provide the optimal solution and optimal value. You may work in groups if you wish.

Solution

1. Let l and h be the volume of light and heavy crude purchased by the refinery, respectively,

measured in hundreds of thousands of barrels. Then we have:

$$\begin{aligned} \min \quad & 35l + 30h \\ \text{s.t.} \quad & 0.3(l+h) \geq 9 \\ & 0.2l + 0.4h \geq 8 \\ & 0.3l + 0.2h \geq 5 \\ & l \geq 0 \\ & h \geq 0 \end{aligned}$$

2. The optimal solution is $l = 0, h = 3M$ with optimal value of \$90M as h is measured in hundreds of thousands.

Problem 2

Consider a school district with N neighborhoods, M schools, and G grades at each school. Each school j has a capacity of C_{jg} for grade g. In each neighborhood i, the student population of grade g is S_{ig}. Finally, the distance of school j from neighborhood i is d_{ij}. Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance travelled by all students. (You may ignore the fact that numbers of students must be integer.)

Solution

Let x_{ijg} be the number of grade g students in neighborhood i assigned to school j. Then we have:

$$\begin{aligned} \min \quad & \sum_{i=0}^{N} \sum_{j=0}^{M} d_{ij} \left(\sum_{g=1}^{G} x_{ijg} \right) \\ \text{s.t.} \quad & \sum_{i=0}^{N} x_{ijg} \leq C_{jg}, \forall j, g \\ & \sum_{j=0}^{M} x_{ijg} = S_{ig}, \forall i, g \\ & x_{ijg} \geq 0, \forall i, j, g \end{aligned}$$

Problem 3

Suppose there are 5 securities available in the World Cup Assets market for open trading at fixed prices and pay-off's; see the table below. Here, for example, Security 1's pay-off is \$1 if either Argentina, Brazil, or Italy wins. The Share Limit represent the maximum number of shares one can purchase, and Price is the current purchasing price per share of each security.

Security	Price π	Share Limit q	Argentina	Brazil	Italy	Germany	France
1	\$ 0.75	10	1	1	1	0	0
2	\$ 0.35	5	0	0	0	1	0
3	\$0.40	10	1	0	1	0	1
4	\$0.95	10	1	1	1	1	0
5	\$0.75	5	0	1	0	1	0

1. Assume that shorting is not allowed, that is, one can only buy shares not sell. Formulate the problem as a linear program to decide how many shares of each security to purchase so as to maximize the worst-case (minimum) profit when the game is finally realized. Solve the LP and provide the optimal solution and value.

2. Now assume there is no share limit (can be ∞) and short is allowed, that is, the decision variable can be both positive (buy) and negative (sell). Reformulate the problem to decide how many shares of each security to purchase so as to maximize the worst-case (minimum) profit when the game is finally realized.

Solution

1. Let x_i be the number of shares purchased of security i and j denote team j.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \max \quad & z - \pi^T x \\ \text{s.t.} \quad & z \leq \sum_i a_{ij} x_i, \forall j \\ & x_i \leq q_i, \forall i \\ & x_i \geq 0, \forall i \end{aligned}$$

2. The LP is the same as above but with the non-negativity and upper bound constraints for x_i removed.

Problem 4

Consider the MDP described in fig 1, this is a modification of the maze run MDP discussed in lecture 2. All actions have zero cost, except the one action from state 4 to state 5 and the red action from state 3 to state 4. Consider the problem of computing the cost-to-go values of the optimal policy for each state. Formulate this as a linear program where the decision variables are the cost-to-go values of the decision states and the discount factor is 0.7.

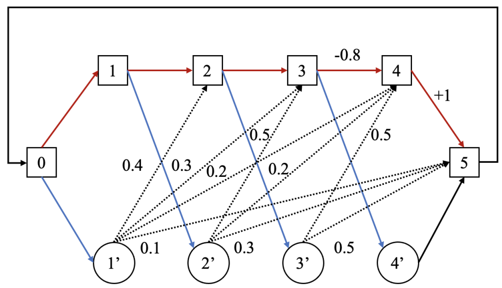


Figure 1: Modified Maze Run

Solution
For $0 \le i \le 5$, let y_i be the cost-to-go for state i. Let $\gamma = 0.7$ be the discount factor. The linear program formulation is:

$$
\begin{aligned}
\max \quad & \sum_{i=0}^{5} y_i \\
\text{s.t.} \quad & y_5 \le \gamma y_0 \\
& y_4 \le 1 + \gamma y_5 \\
& y_3 \le -0.8 + \gamma y_4 \\
& y_3 \le \gamma y_5 \\
& y_2 \le \gamma y_3 \\
& y_2 \le \gamma(0.5 y_4 + 0.5 y_5) \\
& y_1 \le \gamma y_2 \\
& y_1 \le \gamma(0.5 y_3 + 0.2 y_4 + 0.3 y_5) \\
& y_0 \le \gamma y_1 \\
& y_0 \le \gamma(0.4 y_2 + 0.3 y_3 + 0.2 y_4 + 0.1 y_5)
\end{aligned}
$$

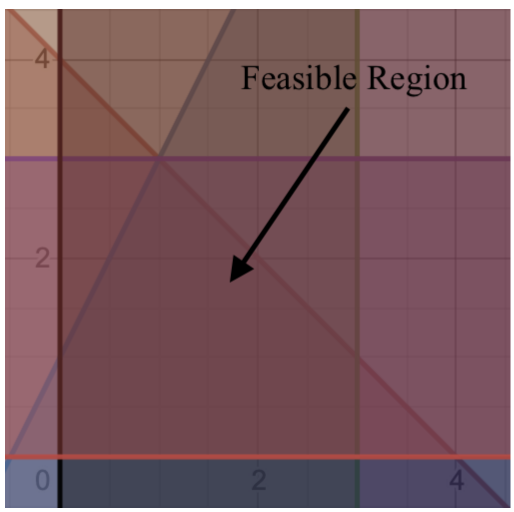
Problem 5

Consider the LP below:

$$
\begin{aligned}
\max \quad & 3x_1 + x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 4 \\
& 2x_1 - x_2 \geq -1 \\
& x_1 \leq 3 \\
& x_2 \leq 3 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

1. Plot the feasible region.
2. Write down the extreme points of the feasible region.
3. Write down the optimal solution and optimal value.

Solution



1.
2. The extreme points are: (0,0), (0,1), (1,3), (3,1), (3,0).
3. The optimal solution is (3,1) with objective function value 10.

Problem 6

Label the followings statements as True or False. For true provide reason and for false either provide reason or a counter example.

1. Consider a linear program with a bounded feasible set. If x is an optimal solution, then it must be an optimal basic feasible solution.

2. The union of a finite number of convex sets is convex.

3. If a linear program has more than one solution, it has infinitely many solutions.

4. Consider the following linear program in standard form:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

   where A has dimension $m \times n$ and its rows are linearly independent. At every optimal solution, at most m variables can be positive.

5. In the linear program of part(4), every corner point of the feasible region has at most $n - m$ adjacent corner points.

Solution

1. False. Consider the following LP, which has a bounded feasible set:

$$\begin{aligned} \max \quad & x + y \\ \text{s.t.} \quad & x + y \leq 4 \\ & x, y \geq 0 \end{aligned}$$

   One optimal solution to this LP is $x = y = 2$. However, it is not a vertex and therefore not a basic feasible solution.

2. False. Consider two sets $A_1 = x_1$ and $A_2 = x_2$, where x_1 and x_2 are distinct points in $\Re$. A_1 and A_2 are convex, but their union $A_1 \cup A_2 = x_1, x_2$ is not.

3. True. Let x_1 and x_2 be two distinct optimal solutions. Then $\lambda x_1 + (1 - \lambda)x_2$ for $0 < \lambda < 1$ is also an optimal solution.

4. False. For $c = 0$, the entire feasible region is the optimal solution set.

5. True. Any corner point is defined by m basic variables. There are $n - m$ nonbasic variables. Every adjacent corner point makes exactly one of these non-basic variables into a new (incoming) basic variable. For any value of the new (incoming) basic variable, the value of the old basic variables are uniquely determined (because A has rank m). Hence there are at most $n - m$ adjacent corner points.