System Dynamics – 3rd Edition
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7

Linear Systems Analysis in the Frequency Domain

7-1 INTRODUCTION

System response to sinusoidal inputs is the major subject of this chapter. We first define the sinusoidal transfer function and explain its use in the steady-state sinusoidal response. Then we treat vibrations in rotating mechanical systems, followed by discussions of vibration isolating problems and dynamic vibration absorbers. Finally, we deal with vibrations in multi-degrees-of-freedom systems.

The outline of the chapter is as follows: Section 7–1 gives introductory material. Section 7–2 begins with forced vibrations of mechanical systems and then derives the sinusoidal transfer function for the dynamic system. Section 7–3 treats vibrations in rotating mechanical systems. Section 7–4 discusses vibration isolation problems that occur in rotating mechanical systems. Here transmissibility for force excitation and that for motion excitation are discussed. Section 7–5 presents a way to reduce vibrations caused by rotating unbalance and treats a dynamic vibration absorber commonly used in industries. Section 7–6 analyzes vibrations in multi-degrees-of-freedom systems and discusses modes of vibration.

7-2 SINUSOIDAL TRANSFER FUNCTION

When a sinusoidal input is applied to a linear system, it will tend to vibrate at its own natural frequency as well as follow the frequency of the input. In the presence of damping, that portion of motion not sustained by the sinusoidal input will gradually die out. As a result, the response at steady state is sinusoidal at the same frequency as the input. The steady-state output differs from the input only in the amplitude and phase angle. Thus the output–input amplitude ratio and the phase angle between the output and input sinusoid are the only two parameters needed to predict the steady-state output of a linear system when the input is a sinusoid. In general, the amplitude ratio and the phase angle depend on the input frequency.

Frequency response. The term *frequency response* refers to the steady-state response of a system to a sinusoidal input. For all frequencies from zero to infinity, the frequency-response characteristics of a system can be completely described by the output–input amplitude ratio and the phase angle between the output and input sinusoid. In this method of systems analysis, we vary the frequency of the input signal over a wide range and study the resulting response. (We shall present detailed discussions of frequency response in Chapter 9.)

Forced vibration without damping. Figure 7–1 illustrates a spring–mass system in which the mass is subjected to a sinusoidal input force $P \sin \omega t$. Let us find the response of the system when it is initially at rest.

If we measure displacement x from the equilibrium position, the equation of motion for the system becomes

$$m\ddot{x} + kx = P \sin \omega t$$

or

$$\ddot{x} + \frac{k}{m} x = \frac{P}{m} \sin \omega t \qquad (7\text{–}1)$$

Note that the solution of this equation consists of the vibration at its own natural frequency (complementary solution) and that at the forcing frequency (particular solution). Thus the solution $x(t)$ can be written as

$$x(t) = \text{complementary solution} + \text{particular solution}$$

Now we shall obtain the solution of Equation (7–1) under the condition that the system is initially at rest. By taking the Laplace transform of Equation (7–1) and using

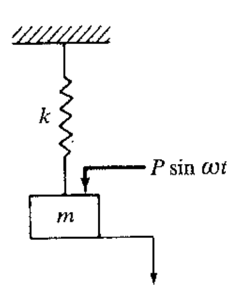

Figure 7–1 Spring–mass system.

the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, we obtain

$$\left(s^2 + \frac{k}{m}\right) X(s) = \frac{P}{m} \frac{\omega}{s^2 + \omega^2}$$

Solving for $X(s)$,

$$X(s) = \frac{P}{m} \frac{\omega}{s^2 + \omega^2} \frac{1}{s^2 + (k/m)}$$

$$= \frac{-P\omega\sqrt{m/k}}{k - m\omega^2} \frac{\sqrt{k/m}}{s^2 + (k/m)} + \frac{P}{k - m\omega^2} \frac{\omega}{s^2 + \omega^2}$$

The inverse Laplace transform of this last equation gives

$$x(t) = -\frac{P\omega\sqrt{m/k}}{k - m\omega^2} \sin\sqrt{\frac{k}{m}}\,t + \frac{P}{k - m\omega^2} \sin \omega t \tag{7-2}$$

This is the complete solution (general solution). The first term is the complementary solution (which does not decay in this system), and the second term is the particular solution. [Note that if we need only a steady-state solution (particular solution) of a stable system the use of the sinusoidal transfer function simplifies the solution. The sinusoidal transfer function is discussed in detail later in this section.]

Let us examine the response of the system, Equation (7–2). As the forcing frequency ω approaches zero, the amplitude of the vibration at its natural frequency $\sqrt{k/m}$ approaches zero and the amplitude of the vibration at the forcing frequency ω approaches P/k. This value P/k is the deflection of the mass that would result if the force P were applied steadily (at zero frequency). That is, P/k is the static deflection. As the frequency ω increases from zero, the denominator of the solution, $k - m\omega^2$, becomes smaller and the amplitudes become larger. As the frequency ω is further increased and becomes equal to the natural frequency of the system, or $\omega = \omega_n = \sqrt{k/m}$, resonance occurs. At resonance the denominator of the solution, $k - m\omega^2$, becomes zero, and the amplitude of vibration will increase without bound. (When the sinusoidal input is applied at the natural frequency and in phase with the motion, that is, in the same direction as the velocity, the input force is actually doing work on the system and is adding to it energy that will appear as an increase in amplitudes.) As ω continues to increase past resonance, the denominator $k - m\omega^2$ becomes negative and assumes increasingly larger values, approaching negative infinity. Therefore, the amplitudes of vibration (at the natural frequency and at the forcing frequency) approach zero from the negative side, starting at negative infinity when $\omega = \omega_n+$. In other words, if ω is below resonance, that part of the vibration at the forcing frequency (particular solution) is in phase with the forcing sinusoid. If ω is above resonance, this vibration becomes 180° out of phase.

Sinusoidal transfer function. The *sinusoidal transfer function* is defined as the transfer function $G(s)$ in which s is replaced by $j\omega$. When only the steady-state solution (particular solution) is wanted, the sinusoidal transfer function $G(j\omega)$ can simplify the solution. In the following discussion we shall consider the behavior of stable linear systems under steady-state conditions, that is, after initial transients have died out. And we shall see that sinusoidal inputs will produce sinusoidal outputs in steady state with the amplitude and phase angle at each frequency ω determined by the magnitude and angle of $G(j\omega)$, respectively.

Deriving steady-state output to sinusoidal input. We shall show how the frequency-response characteristics of a stable system can be derived directly from the sinusoidal transfer function. For the linear system $G(s)$ shown in Figure 7–2, the input and output are denoted by $p(t)$ and $x(t)$, respectively. The input $p(t)$ is sinusoidal and is given by

$$p(t) = P \sin \omega t$$

We shall show that the output $x(t)$ at steady state is given by

$$x(t) = |G(j\omega)| P \sin(\omega t + \phi)$$

where $|G(j\omega)|$ and ϕ are the magnitude and angle of $G(j\omega)$, respectively.

Suppose that the transfer function $G(s)$ can be written as a ratio of two polynomials in s; that is,

$$G(s) = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + s_1)(s + s_2) \cdots (s + s_n)}$$

The Laplace transformed output $X(s)$ is

$$X(s) = G(s)P(s) \tag{7–3}$$

where $P(s)$ is the Laplace transform of the input $p(t)$.

Let us limit our discussion only to stable systems. For such systems, the real parts of the $-s_i$ are negative. The steady-state response of a stable linear system to a sinusoidal input does not depend on the initial conditions, and so they can be ignored.

If $G(s)$ has only distinct poles, then the partial-fraction expansion of Equation (7–3) yields

$$X(s) = G(s)\frac{P\omega}{s^2 + \omega^2}$$

$$= \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s + s_1} + \frac{b_2}{s + s_2} + \cdots + \frac{b_n}{s + s_n} \tag{7–4}$$

where a and b_i (where $i = 1, 2, \ldots, n$) are constants and $\bar{a}$ is the complex conjugate of a. The inverse Laplace transform of Equation (7–4) gives

$$x(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1e^{-s_1t} + b_2e^{-s_2t} + \cdots + b_ne^{-s_nt}$$

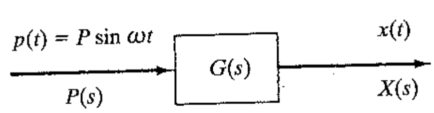

Figure 7–2 Linear system.

For a stable system, as t approaches infinity, the terms $e^{-s_1 t}, e^{-s_2 t}, \ldots, e^{-s_n t}$ approach zero, since $-s_1, -s_2, \ldots, -s_n$ have negative real parts. Thus all terms on the right-hand side of this last equation, except the first two, drop out at steady state.

If $G(s)$ involves k multiple poles s_j, then $x(t)$ will involve such terms as $t^h e^{-s_j t}$ (where $h = 0, 1, \ldots, k-1$). Since the real part of the $-s_j$ is negative for a stable system, the terms $t^h e^{-s_j t}$ approach zero as t approaches infinity.

Regardless of whether the system involves multiple poles, the steady-state response thus becomes

$$x(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} \tag{7–5}$$

where the constants a and $\bar{a}$ can be evaluated from Equation (7–4).

$$a = G(s)\frac{P\omega}{s^2+\omega^2}(s+j\omega)\Big|_{s=-j\omega} = -\frac{P}{2j}G(-j\omega)$$

$$\bar{a} = G(s)\frac{P\omega}{s^2+\omega^2}(s-j\omega)\Big|_{s=j\omega} = \frac{P}{2j}G(j\omega)$$

(Note that $\bar{a}$ is the complex conjugate of a.) Referring to Figure 7–3, we can write

$$\begin{aligned} G(j\omega) &= G_x + jG_y \\ &= |G(j\omega)| \cos\phi + j|G(j\omega)| \sin\phi \\ &= |G(j\omega)| (\cos\phi + j\sin\phi) \\ &= |G(j\omega)| e^{j\phi} \end{aligned}$$

(Note that $\underline{/G(j\omega)} = \underline{/e^{j\phi}} = \phi$.) Similarly,

$$G(-j\omega) = |G(-j\omega)|e^{-j\phi} = |G(j\omega)|e^{-j\phi}$$

It follows that

$$a = -\frac{P}{2j}|G(j\omega)|e^{-j\phi}$$

$$\bar{a} = \frac{P}{2j}|G(j\omega)|e^{j\phi}$$

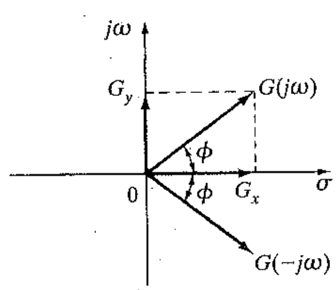

Figure 7–3 Complex function and its complex conjugate.

Then Equation (7–5) can be written as

$$\begin{aligned} x(t) &= |G(j\omega)|\,P\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \\ &= |G(j\omega)|\,P\sin(\omega t+\phi) \\ &= X\sin(\omega t+\phi) \qquad (7\text{–}6) \end{aligned}$$

where $X = |G(j\omega)|P$ and $\phi = \underline{/G(j\omega)}$. We see that a stable linear system subjected to a sinusoidal input will, at steady state, have a sinusoidal output of the same frequency as the input. But the amplitude and phase angle of the output will, in general, differ from the input's. In fact, the output's amplitude is given by the product of the amplitude of the input and $|G(j\omega)|$, whereas the phase angle differs from that of the input by the amount $\phi = \underline{/G(j\omega)}$.

On the basis of the preceding analysis, we are able to derive the following important result. For sinusoidal inputs,

$$|G(j\omega)| = \left|\frac{X(j\omega)}{P(j\omega)}\right| = \begin{matrix}\text{amplitude ratio of the output}\\ \text{sinusoid to the input sinusoid}\end{matrix} \qquad (7\text{–}7)$$

$$\begin{aligned} \underline{/G(j\omega)} &= \underline{\bigg/\frac{X(j\omega)}{P(j\omega)}} = \tan^{-1}\left[\frac{\text{imaginary part of } G(j\omega)}{\text{real part of } G(j\omega)}\right] \\ &= \begin{matrix}\text{phase shift of the output sinusoid}\\ \text{with respect to the input sinusoid}\end{matrix} \qquad (7\text{–}8) \end{aligned}$$

Thus the steady-state response characteristics of a linear system to a sinusoidal input can be found directly from $G(j\omega)$, the ratio of $X(j\omega)$ to $P(j\omega)$.

Note that the sinusoidal transfer function $G(j\omega)$ is a complex quantity that can be represented by the magnitude and phase angle with frequency ω as a parameter. To characterize a linear system completely by the frequency-response curves, we must specify both the amplitude ratio and the phase angle as a function of the frequency ω.

Comments. Equation (7–6) is valid only if $G(s) = X(s)/P(s)$ is a stable system, that is, if all poles of $G(s)$ lie in the left half s plane. If a pole is at the origin and/or poles of $G(s)$ lie on the $j\omega$ axis (any poles on the $j\omega$ axis, except at the origin, must occur as a pair of complex conjugates), the output $x(t)$ may be obtained by taking the inverse Laplace transform of the equation

$$X(s) = G(s)P(s) = G(s)\frac{P\omega}{s^2+\omega^2}$$

or

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[G(s)\frac{P\omega}{s^2+\omega^2}\right]$$

Note that if a pole or poles of $G(s)$ lie in the right half s plane the system is unstable and the response grows indefinitely. There is no steady state for such an unstable system.

Example 7–1

Consider the transfer-function system

$$\frac{X(s)}{P(s)} = G(s) = \frac{1}{Ts + 1}$$

For the sinusoidal input $p(t) = P \sin \omega t$, what is the steady-state output $x(t)$?

Substituting $j\omega$ for s in $G(s)$ yields

$$G(j\omega) = \frac{1}{Tj\omega + 1}$$

The output–input amplitude ratio is

$$|G(j\omega)| = \frac{1}{\sqrt{T^2\omega^2 + 1}}$$

whereas the phase angle ϕ is

$$\phi = \underline{/G(j\omega)} = -\tan^{-1} T\omega$$

So for the input $p(t) = P \sin \omega t$ the steady-state output $x(t)$ can be found as

$$x(t) = \frac{P}{\sqrt{T^2\omega^2 + 1}} \sin(\omega t - \tan^{-1} T\omega) \tag{7–9}$$

From this equation we see that, for small ω, the amplitude of the output $x(t)$ is almost equal to the amplitude of the input. For large ω, the amplitude of the output is small and almost inversely proportional to ω. The phase angle is 0° at $\omega = 0$ and approaches $-90°$ as ω increases indefinitely.

Example 7–2

Suppose that a sinusoidal force $p(t) = P \sin \omega t$ is applied to the mechanical system shown in Figure 7–4. Assuming that displacement x is measured from the equilibrium position, find the steady-state output.

The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = p(t)$$

The Laplace transform of this equation, assuming zero initial conditions, is

$$(ms^2 + bs + k)X(s) = P(s)$$

where $X(s) = \mathscr{L}[x(t)]$ and $P(s) = \mathscr{L}[p(t)]$. (Note that the initial conditions do not affect the steady-state output and so can be assumed zero.) The transfer function be-

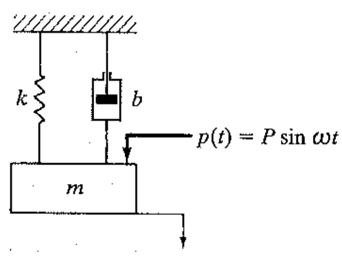

Figure 7–4 Mechanical system.

tween displacement $X(s)$ and input force $P(s)$ is, therefore, obtained as

$$\frac{X(s)}{P(s)} = G(s) = \frac{1}{ms^2 + bs + k}$$

Since the input is a sinusoidal function $p(t) = P \sin \omega t$, we can use the sinusoidal transfer function to obtain the steady-state solution. The sinusoidal transfer function is

$$\frac{X(j\omega)}{P(j\omega)} = G(j\omega) = \frac{1}{-m\omega^2 + bj\omega + k} = \frac{1}{(k - m\omega^2) + jb\omega}$$

Referring to Equation (7–6), the steady-state output $x(t)$ can be written

$$x(t) = |G(j\omega)| P \sin(\omega t + \phi)$$

where

$$|G(j\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

and

$$\phi = \underline{/G(j\omega)} = \underline{\Big/ \frac{1}{(k - m\omega^2) + jb\omega}} = -\tan^{-1}\frac{b\omega}{k - m\omega^2}$$

Thus

$$x(t) = \frac{P}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \sin\left(\omega t - \tan^{-1}\frac{b\omega}{k - m\omega^2}\right)$$

Since $k/m = \omega_n^2$ and $b/k = 2\zeta/\omega_n$, this equation can be written

$$x(t) = \frac{x_{st}}{\sqrt{[1 - (\omega^2/\omega_n^2)]^2 + (2\zeta\omega/\omega_n)^2}} \sin\left[\omega t - \tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - (\omega^2/\omega_n^2)}\right] \qquad (7\text{–}10)$$

where $x_{st} = P/k$ is the static deflection.

By writing the amplitude of $x(t)$ as X, we find that the amplitude ratio X/x_{st} is

$$\frac{X}{x_{st}} = \frac{1}{\sqrt{[1 - (\omega^2/\omega_n^2)]^2 + (2\zeta\omega/\omega_n)^2}}$$

and the phase shift ϕ is

$$\phi = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - (\omega^2/\omega_n^2)}$$

7–3 VIBRATIONS IN ROTATING MECHANICAL SYSTEMS

Vibration is, in general, undesirable because it may cause parts to break down, generate noise, transmit forces to the foundation, and so on. To reduce the amount of force transmitted to the foundation as a result of a machine's vibration (force isolation) as much as possible, machines are usually mounted on vibration isolators that consist of springs and dampers. Similarly, to reduce the amount of motion transmitted to a delicate instrument by the motion of the foundation (motion isolation),