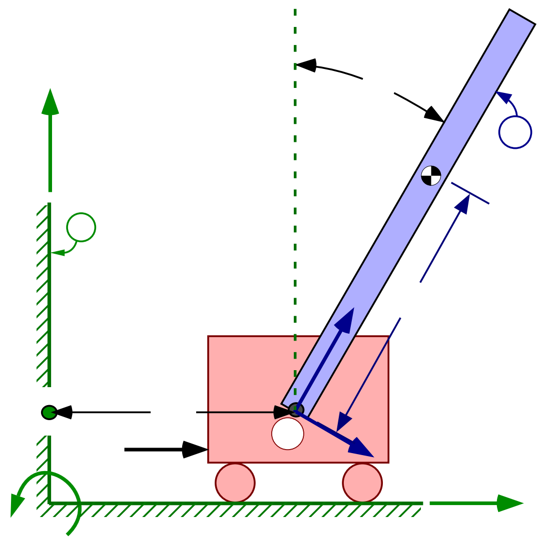
# Chapter 11

# Dynamics: Inverted pendulum on a cart

The figure to the right shows a rigid inverted pendulum $B$ attached by a frictionless revolute joint to a cart $A$ (modeled as a particle). The cart $A$ slides on a horizontal frictionless track that is fixed in a Newtonian reference frame $N$. Right-handed sets of unit vectors $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$ and $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ are fixed in $N$ and $B$ respectively, with:

- $\mathbf{n}_x$ horizontal and to the right
- $\mathbf{n}_y$ vertically upward
- $\mathbf{n}_z$=$\mathbf{b}_z$ parallel to the axis of rotation of $B$ in $N$
- $\mathbf{b}_y$ directed from $A$ to the distal end of $B$
- $\mathbf{b}_x = \mathbf{b}_y \times \mathbf{b}_z$



The identifiers in the following table are useful while forming equations of motion for this system. ***Complete*** the figure to the right by adding the identifiers $N$, $A$, $B$, $B_{cm}$, $L$, $F_c$, $x$, $\theta$, $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$, $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$.

| Quantity | Symbol | Value |
|---|---|---|
| Mass of $A$ | $m^A$ | 10.0 kg |
| Mass of $B$ | $m^B$ | 1.0 kg |
| Distance from $A$ to $B_{cm}$ (the mass center of $B$) | $L$ | 0.5 m |
| Central moment of inertia of $B$ for $\mathbf{b}_z$ | $I_{zz}$ | 0.08333 kg∗m$^2$ |
| Earth's sea-level gravitational constant | $g$ | 9.81 m/sec$^2$ |
| Feedback-control force applied to $A$ in $\mathbf{n}_x$ direction | $F_c$ | specified |
| Distance from $N_o$ (a point fixed in $N$) to $A$ | $x$ | variable |
| Angle between the local vertical and the long axis of $B$ | $\theta$ | variable |

## 11.1 Kinematics (space and time)

Kinematics is the study of the relationship between space and time, and is independent of the influence of mass or forces. The kinematic quantities normally needed for dynamic analysis are listed below. In most circumstances, it is efficient to form rotation matrices, angular velocities, and angular accelerations before position vectors, velocities, and accelerations.

| Kinematic Quantity | Quantities needed for analyzing the inverted pendulum on a cart |
|---|---|
| Rotation matrix | ${}^{b}R^{n}$, the rotation matrix relating $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ and $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$ |
| Angular velocity | ${}^{N}\boldsymbol{\omega}^{B}$, the angular velocity of $B$ in $N$ |
| Angular acceleration | ${}^{N}\boldsymbol{\alpha}^{B}$, the angular acceleration of $B$ in $N$ |
| Position vectors | $\mathbf{r}^{A/N_o}$ and $\mathbf{r}^{B_{cm}/A}$, the position vector of $A$ from $N_o$ and of $B_{cm}$ from $A$ |
| Velocity | ${}^{N}\mathbf{v}^{A}$ and ${}^{N}\mathbf{v}^{B_{cm}}$, the velocity of $A$ in $N$ and the velocity of $B_{cm}$ in $N$ |
| Acceleration | ${}^{N}\mathbf{a}^{A}$ and ${}^{N}\mathbf{a}^{B_{cm}}$, the acceleration of $A$ in $N$ and the acceleration of $B_{cm}$ in $N$ |

## 11.2 Rotation matrices

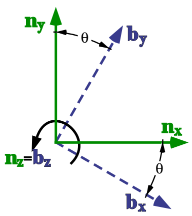
The orientation of a set of mutually-perpendicular right-handed unit vectors $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ in a second set of mutually-perpendicular right-handed unit vectors $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$ is frequently stored in a 3×3 ***rotation matrix*** denoted ${}^{b}R^{n}$. The elements of ${}^{b}R^{n}$ are defined as ${}^{b}R^{n}_{ij} \triangleq \mathbf{b_i} \cdot \mathbf{n_j} \quad (i, j = x, y, z)$

All rotation matrices are ***orthogonal***, which means that its inverse is equal to its transpose and it can be written as a table read horizontally or vertically.

| ${}^{b}R^{n}$ | $\mathbf{n}_x$ | $\mathbf{n}_y$ | $\mathbf{n}_z$ |
|---|---|---|---|
| $\mathbf{b}_x$ | ${}^{b}R^{n}_{xx}$ | ${}^{b}R^{n}_{xy}$ | ${}^{b}R^{n}_{xz}$ |
| $\mathbf{b}_y$ | ${}^{b}R^{n}_{yx}$ | ${}^{b}R^{n}_{yy}$ | ${}^{b}R^{n}_{yz}$ |
| $\mathbf{b}_z$ | ${}^{b}R^{n}_{zx}$ | ${}^{b}R^{n}_{zy}$ | ${}^{b}R^{n}_{zz}$ |

### Rotation matrix example

The system has two sets of mutually-perpendicular right-handed unit vectors, namely $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ and $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$. These two sets of vectors are drawn in a geometrically suggestive way below. One way to calculate the first row of the ${}^{b}R^{n}$ rotation matrix is by expressing $\mathbf{b}_x$ in terms of $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$, and then filling in the first row of ${}^{b}R^{n}$ as shown below. Similarly, the second and third rows of ${}^{b}R^{n}$ are calculated by expressing $\mathbf{b}_y$ and $\mathbf{b}_z$ in terms of $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$, and filling in the second and third rows of ${}^{b}R^{n}$.




$$\begin{aligned} \mathbf{b}_x &= \cos(\theta)\,\mathbf{n}_x - \sin(\theta)\,\mathbf{n}_y \\ \mathbf{b}_y &= \sin(\theta)\,\mathbf{n}_x + \cos(\theta)\,\mathbf{n}_y \\ \mathbf{b}_z &= \mathbf{n}_z \end{aligned}$$

| ${}^{b}R^{n}$ | $\mathbf{n}_x$ | $\mathbf{n}_y$ | $\mathbf{n}_z$ |
|---|---|---|---|
| $\mathbf{b}_x$ | $\cos(\theta)$ | $-\sin(\theta)$ | 0 |
| $\mathbf{b}_y$ | $\sin(\theta)$ | $\cos(\theta)$ | 0 |
| $\mathbf{b}_z$ | 0 | 0 | 1 |

## 11.3 Angular velocity

As shown in equation (1), the angular velocity of a reference frame $B$ in a reference frame $N$ can be calculated directly from the rotation matrix that relates $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ to $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$ and its time-derivative.

Since equation (1) contains $\dot{R}_{ij}\ (i, j = x, y, z)$ it is clear that angular velocity is a measure of the time-rate of change of orientation.

$$R_{ij} \triangleq \mathbf{b_i} \cdot \mathbf{n_j} \quad (i, j = x, y, z)$$

$$\begin{aligned} {}^{N}\boldsymbol{\omega}^{B} = &\left(R_{xz}\dot{R}_{xy} + R_{yz}\dot{R}_{yy} + R_{zz}\dot{R}_{zy}\right)\mathbf{b}_x \\ + &\left(R_{yx}\dot{R}_{yz} + R_{zx}\dot{R}_{zz} + R_{xx}\dot{R}_{xz}\right)\mathbf{b}_y \\ + &\left(R_{zy}\dot{R}_{zx} + R_{xy}\dot{R}_{xx} + R_{yy}\dot{R}_{yx}\right)\mathbf{b}_z \end{aligned} \tag{1}$$

### 11.3.1 Simple angular velocity

Although equation (1) is a general relationship between the rotation matrix and angular velocity, it is non-intuitive.[1] Since angular velocity is complicated, most textbooks define ***simple angular velocity***, which is useful for ***two-dimensional analysis***. The simple angular velocity of $B$ in $N$ is calculated as

$$ {}^{N}\boldsymbol{\omega}^{B} \underset{\text{(simple)}}{=} \pm\,\dot{\theta}\boldsymbol{\lambda} \tag{2}$$

where $\boldsymbol{\lambda}$ is a vector ***fixed***[2] in both $N$ and $B$. The sign of $\dot{\theta}\boldsymbol{\lambda}$ is determined by the right-hand rule. If increasing $\theta$ causes a right-hand rotation of $B$ in $N$ about $^{+}\boldsymbol{\lambda}$, the sign is positive, otherwise it is negative.

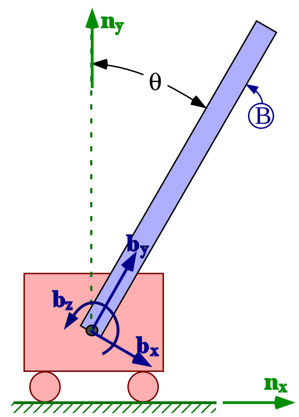
[1] One of the major obstacles in three-dimensional kinematics is properly calculating angular velocity.

[2] A vector $\boldsymbol{\lambda}$ is said to be fixed in reference frame $B$ if its magnitude is constant and its direction does not change in $B$.

The following is a step-by-step process for calculating a simple angular velocity ${}^N\boldsymbol{\omega}^B$:

- Identify a unit vector $\boldsymbol{\lambda}$ that is fixed in both $N$ and $B$
- Identify a vector $\mathbf{n}_\perp$ that is fixed in $N$ and perpendicular to $\boldsymbol{\lambda}$
- Identify a vector $\mathbf{b}_\perp$ that is fixed in $B$ and perpendicular to $\boldsymbol{\lambda}$
- Identify the angle $\theta$ between $\mathbf{n}_\perp$ and $\mathbf{b}_\perp$ and calculate its time-derivative
- Use the right-hand rule to determine the sign of $\boldsymbol{\lambda}$. In other words, point the four fingers of your right hand in the direction of $\mathbf{n}_\perp$, and then curl them in the direction of $\mathbf{b}_\perp$. If you thumb points in the direction of $\boldsymbol{\lambda}$, the sign of $\boldsymbol{\lambda}$ is positive, otherwise it is negative.

### 11.3.2 Simple angular velocity example




The figure to the right has two reference frames, $B$ and $N$. Since $\mathbf{b}_z$ is fixed in both $B$ and $N$, the angular velocity of $B$ in $N$ is a simple angular velocity that can be written as

$$ {}^N\boldsymbol{\omega}^B \;=\; \text{-}\dot{\theta}\,\mathbf{b}_z $$

The following step-by-step process was used to calculate ${}^N\boldsymbol{\omega}^B$:

- $\mathbf{b}_z$ is a unit vector that is fixed in both $N$ and $B$
- $\mathbf{n}_y$ is fixed in $N$ and perpendicular to $\mathbf{b}_z$
- $\mathbf{b}_y$ is fixed in $B$ and perpendicular to $\mathbf{b}_z$
- $\theta$ is the angle between $\mathbf{n}_y$ and $\mathbf{b}_y$, and $\dot{\theta}$ is its time-derivative
- After pointing the four fingers of your right hand in the direction of $\mathbf{n}_y$ and curling them in the direction of $\mathbf{b}_y$, your thumb points in the -$\mathbf{b}_z$ direction. Hence, the sign of $\mathbf{b}_z$ is negative.

### 11.3.3 Angular velocity and vector differentiation

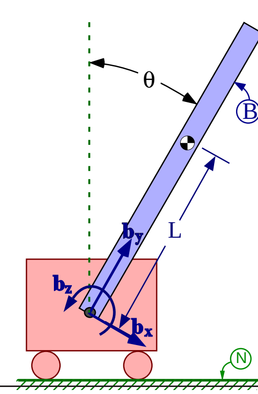
The ***golden rule for vector differentiation*** calculates $\frac{{}^N d\mathbf{v}}{dt}$, the ordinary-derivative of $\mathbf{v}$ with respect to $t$ in $N$, in terms of the following:

- Two reference frames $N$ and $B$
- ${}^N\boldsymbol{\omega}^B$, the angular velocity of $B$ in $N$
- $\mathbf{v}$, ***any*** vector that is a function of a single scalar variable $t$
- $\frac{{}^B d\mathbf{v}}{dt}$, the ordinary-derivative of $\mathbf{v}$ with respect to $t$ in $B$

$$ \boxed{\frac{{}^N d\mathbf{v}}{dt} \;=\; \frac{{}^B d\mathbf{v}}{dt} \;+\; {}^N\boldsymbol{\omega}^B \times \mathbf{v}} \qquad (3) $$

Equation (3) is one of the ***most important formulas in kinematics*** because $\mathbf{v}$ can be ***any*** vector, e.g., a unit vector, a position vector, a velocity vector, a linear/angular acceleration vector, a linear/angular momentum vector, or a force or torque vector.

### 11.3.4 Angular velocity and vector differentiation example




An efficient way to calculate the time-derivative in $N$ of $L\mathbf{b}_y$ is as follows:

$$ \begin{aligned} \frac{{}^N d\,(L\mathbf{b}_y)}{dt} \;&\underset{(3)}{=}\; \frac{{}^B d\,(L\mathbf{b}_y)}{dt} \;+\; {}^N\boldsymbol{\omega}^B \times L\mathbf{b}_y \\ &=\; \mathbf{0} \;+\; \left(\text{-}\dot{\theta}\mathbf{b}_z\right) \times L\mathbf{b}_y \\ &=\; \dot{\theta}\,L\,\mathbf{b}_x \end{aligned} $$

## 11.4 Angular acceleration

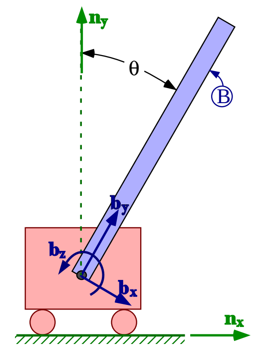
As shown in equation (4), the angular acceleration of a reference frame $B$ in a reference frame $N$ is defined as the time-derivative in $N$ of ${}^N\boldsymbol{\omega}^B$.

${}^N\boldsymbol{\alpha}^B$ also *happens* to be equal to the time-derivative in **B** of ${}^N\boldsymbol{\omega}^B$.

Note: Employ this useful property of angular acceleration when it is easier to compute $\frac{{}^{\mathbf{B}}d\,{}^N\boldsymbol{\omega}^B}{dt}$ than it is to compute $\frac{{}^{\mathbf{N}}d\,{}^N\boldsymbol{\omega}^B}{dt}$.

$$ {}^N\boldsymbol{\alpha}^B \triangleq \frac{{}^Nd\,{}^N\boldsymbol{\omega}^B}{dt} = \frac{{}^Bd\,{}^N\boldsymbol{\omega}^B}{dt} \qquad (4) $$

### Angular acceleration example

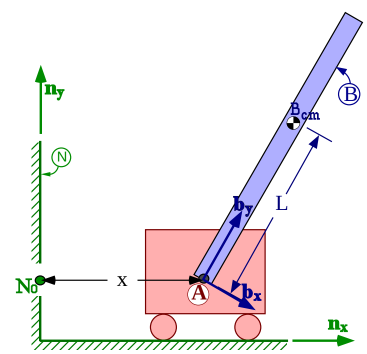



The figure to the right has two reference frames, $B$ and $N$. Although the angular acceleration of $B$ in $N$ is defined as ${}^N\boldsymbol{\alpha}^B \triangleq \frac{{}^Nd\,{}^N\boldsymbol{\omega}^B}{dt}$, it is more easily calculated with the alternate definition, i.e.,

$$ {}^N\boldsymbol{\alpha}^B = \frac{{}^Bd\,{}^N\boldsymbol{\omega}^B}{dt} = \frac{{}^Bd\,(-\dot{\theta}\,\mathbf{b}_z)}{dt} = -\ddot{\theta}\,\mathbf{b}_z $$

## 11.5 Position vectors

A point's ***position vector*** characterizes its location from another point.[3]




The figure on the right shows three points $N_o$, $A$, and $B_{cm}$.
By **inspection**, one can determine:

- $\mathbf{r}^{A/N_o} = x\,\mathbf{n}_x$     ($A$'s position vector from $N_o$)
- $\mathbf{r}^{B_{cm}/A} = L\,\mathbf{b}_y$     ($B_{cm}$'s position vector from $A$)

$B_{cm}$'s position vector from $N_o$ is computed with vector addition, i.e.,

$$ \mathbf{r}^{B_{cm}/N_o} = \mathbf{r}^{B_{cm}/A} + \mathbf{r}^{A/N_o} = L\,\mathbf{b}_y + x\,\mathbf{n}_x $$

## 11.6 Velocity

The velocity of a point $B_{cm}$ in a reference frame $N$ is denoted ${}^N\mathbf{v}^{B_{cm}}$ and is defined as the time-derivative in $N$ of $\mathbf{r}^{B_{cm}/N_o}$ ($B_{cm}$'s position vector from $N_o$). Note: Point $N_o$ is ***any*** point ***fixed*** in $N$.

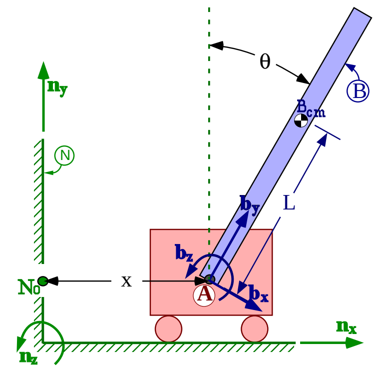
$$ {}^N\mathbf{v}^{B_{cm}} \triangleq \frac{{}^Nd\,\mathbf{r}^{B_{cm}/N_o}}{dt} \qquad (5) $$

[3] Since a position vector locates a *point* from another *point* and because a body contains an infinite number of points, a body cannot be uniquely located with a position vector. In short, a body does **not** have a position vector.

### Velocity example

The figure to the right shows a point $B_{cm}$ moving in reference frame $N$. Differentiating $B_{cm}$'s position vector from $N_o$ yields $B_{cm}$'s velocity in $N$.

$$
\begin{aligned}
{}^{N}\mathbf{v}^{B_{cm}} &\triangleq \frac{{}^{N}d\,\mathbf{r}^{B_{cm}/N_o}}{dt} \\
&= \frac{{}^{N}d\,(x\,\mathbf{n}_\mathrm{x} \;+\; L\,\mathbf{b}_\mathrm{y})}{dt} \\
&= \frac{{}^{N}d\,(x\,\mathbf{n}_\mathrm{x})}{dt} + \frac{{}^{N}d\,(L\,\mathbf{b}_\mathrm{y})}{dt} \\
&= \dot{x}\,\mathbf{n}_\mathrm{x} + \frac{{}^{B}d\,(L\,\mathbf{b}_\mathrm{y})}{dt} + {}^{N}\boldsymbol{\omega}^{B} \times L\,\mathbf{b}_\mathrm{y} \\
&= \dot{x}\,\mathbf{n}_\mathrm{x} + \mathbf{0} + \text{-}\dot{\theta}\,\mathbf{b}_\mathrm{z} \times L\,\mathbf{b}_\mathrm{y} \\
&= \dot{x}\,\mathbf{n}_\mathrm{x} + \dot{\theta}\,L\,\mathbf{b}_\mathrm{x}
\end{aligned}
$$




## 11.7 Acceleration

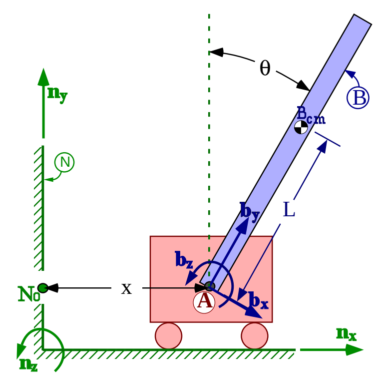
The acceleration of a point $B_{cm}$ in a reference frame $N$ is denoted ${}^{N}\mathbf{a}^{B_{cm}}$ and is defined as the time-derivative in $N$ of ${}^{N}\mathbf{v}^{B_{cm}}$ ($B_{cm}$'s velocity in $N$).

$$
{}^{N}\mathbf{a}^{B_{cm}} \triangleq \frac{{}^{N}d\;{}^{N}\mathbf{v}^{B_{cm}}}{dt} \qquad (6)
$$

### Acceleration example

The figure to the right shows a point $B_{cm}$ moving in reference frame $N$. Differentiating $B_{cm}$'s velocity in $N$ yields $B_{cm}$'s acceleration in $N$. [a]

$$
\begin{aligned}
{}^{N}\mathbf{a}^{B_{cm}} &\triangleq \frac{{}^{N}d\;{}^{N}\mathbf{v}^{B_{cm}}}{dt} \\
&= \frac{{}^{N}d\,(\dot{x}\,\mathbf{n}_\mathrm{x} \;+\; \dot{\theta}\,L\,\mathbf{b}_\mathrm{x})}{dt} \\
&= \frac{{}^{N}d\,(\dot{x}\,\mathbf{n}_\mathrm{x})}{dt} + \frac{{}^{N}d\,(\dot{\theta}\,L\,\mathbf{b}_\mathrm{x})}{dt} \\
&= \ddot{x}\,\mathbf{n}_\mathrm{x} + \frac{{}^{B}d\,(\dot{\theta}\,L\mathbf{b}_\mathrm{x})}{dt} + {}^{N}\boldsymbol{\omega}^{B} \times (\dot{\theta}\,L\,\mathbf{b}_\mathrm{x}) \\
&= \ddot{x}\,\mathbf{n}_\mathrm{x} + \ddot{\theta}\,L\;\mathbf{b}_\mathrm{x} + (\text{-}\dot{\theta}\,\mathbf{b}_\mathrm{z}) \times (\dot{\theta}\,L\,\mathbf{b}_\mathrm{x}) \\
&= \ddot{x}\,\mathbf{n}_\mathrm{x} + \ddot{\theta}\,L\,\mathbf{b}_\mathrm{x} + \text{-}\dot{\theta}^2\,L\,\mathbf{b}_\mathrm{y}
\end{aligned}
$$




[a] There are certain acceleration terms that have special names, e.g., "Coriolis", "centripetal", and "tangential". Knowing the names of acceleration terms is *significantly* less important than knowing how to correctly form the acceleration.

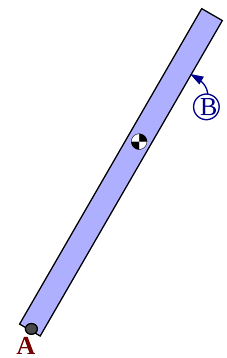
## 11.8 Mass distribution

Mass distribution is the study of mass, center of mass, and inertia properties of systems components. One way to experimentally determine an object's mass distribution properties is to measure mass with a scale, measure center of mass by hanging the object and drawing vertical lines, and measure moments of inertia by timing periods of oscillation. Alternately, by knowing the geometry and material type, it is possible to calculate the mass, center of mass, and inertia properties. These calculations are automatically performed in CAD and motion programs such as SolidWorks, Solid Edge, Inventor, Pro/E, Working Model, MSC.visualNastran 4D, MSC.Adams, etc. In general, the quantities needed for dynamic analysis are:

- the mass of each particle, e.g., the mass of particle $A$ is $m^A$
- the mass of each body, e.g., the mass of body $B$ is $m^B$
- the central inertia dyadic of each body. Since $B$ has a *simple* angular velocity in $N$, $I_{zz}$ (the moment of inertia of $B$ about $B_{cm}$ for the $\mathbf{b}_\text{z}$ axis) is sufficient for this analyses.

## 11.9 Contact and distance forces

One way to analyze forces is to use a free-body diagram to isolate a single body and draw all the forces that act on it. Use the figure on the right to draw all the contact and distance forces on the cart $A$ and pendulum $B$.[a]

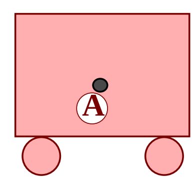



| Quantity | Description |
|---|---|
| $F_c$ | $\mathbf{n}_\text{x}$ measure of control force applied to $A$ |
| $m^A\,g$ | -$\mathbf{n}_\text{y}$ measure of local gravitational force on $A$ |
| $N$ | $\mathbf{n}_\text{y}$ measure of the resultant normal force on $A$ |
| $R_x$ | $\mathbf{n}_\text{x}$ measure of the force on $B$ from $A$ |
| $R_y$ | $\mathbf{n}_\text{y}$ measure of the force on $B$ from $A$ |
| $m^B\,g$ | -$\mathbf{n}_\text{y}$ measure of local gravitational force on $B$ |

The resultant forces on $A$ and $B$ are

$$\mathbf{F}^A = (F_c - R_x)\,\mathbf{n}_\text{x} + (N - m^A\,g - R_y)\,\mathbf{n}_\text{y}$$

$$\mathbf{F}^B = R_x\,\mathbf{n}_\text{x} + (R_y - m^B\,g)\,\mathbf{n}_\text{y}$$




[a] When a force on a point is applied by another point that is part of the system being considered, it is conventional to use action/reaction to minimize the number of unknowns. Notice that the force on $A$ from $B$ is treated using action/reaction, whereas the force on $A$ from $N$ is not.

## 11.10 Moments

The moment of all forces on $B$ about $B_{cm}$ is[4]

$$\begin{aligned}
\mathbf{M}^{B/B_{cm}} &= \mathbf{r}^{A/B_{cm}} \times (R_x\,\mathbf{n}_\text{x} + R_y\,\mathbf{n}_\text{y}) \;+\; \mathbf{r}^{B_{cm}/B_{cm}} \times \left(\text{-}m^B\,g\,\mathbf{n}_\text{y}\right) \\
&= \text{-}L\,\mathbf{b}_\text{y} \times (R_x\,\mathbf{n}_\text{x} + R_y\,\mathbf{n}_\text{y}) \\
&= \text{-}L\,R_x\,(\mathbf{b}_\text{y} \times \mathbf{n}_\text{x}) \;+\; \text{-}L\,R_y\,(\mathbf{b}_\text{y} \times \mathbf{n}_\text{y}) \\
&= [L\cos(\theta)R_x \;-\; L\sin(\theta)R_y]\;\mathbf{b}_\text{z}
\end{aligned}$$

[4] Use the rotation table to calculate the cross-products $(\mathbf{b}_\text{y} \times \mathbf{n}_\text{x})$ and $(\mathbf{b}_\text{y} \times \mathbf{n}_\text{y})$.

## 11.11 Newton's law of motion

Newton's law of motion for particle $A$ is $\mathbf{F}^A = m^A * {}^N\mathbf{a}^A$, where: $\mathbf{F}^A$ is the resultant of all forces on $A$; $m^A$ is the mass of $A$; and ${}^N\mathbf{a}^A$ is the acceleration of $A$ in $N$. Substituting into Newton's law produces

$$(F_c - R_x)\,\mathbf{n}_x \;+\; (N - m^A g - R_y)\,\mathbf{n}_y \;=\; m^A\,\ddot{x}\,\mathbf{n}_x$$

Dot-multiplication with $\mathbf{n}_x$ gives:

$$F_c \;-\; R_x \;=\; m^A\,\ddot{x}$$

Dot-multiplication with $\mathbf{n}_y$ gives:

$$N \;-\; m^A\,g \;-\; R_y \;=\; 0$$

Similarly, Newton's law of motion for body $B$ is $\mathbf{F}^B = m^B * {}^N\mathbf{a}^{B_{cm}}$, where: $\mathbf{F}^B$ is the resultant of all forces on $B$; $m^B$ is the mass of $B$; and ${}^N\mathbf{a}^{B_{cm}}$ is the acceleration of $B$'s mass center in $N$. This produces

$$R_x\,\mathbf{n}_x \;+\; (R_y - m^B g)\,\mathbf{n}_y \;=\; m^B\left(\ddot{x}\,\mathbf{n}_x \;+\; \ddot{\theta}\,L\,\mathbf{b}_x \;-\; \dot{\theta}^2\,L\,\mathbf{b}_y\right)$$

Dot-multiplication with $\mathbf{n}_x$ gives:[a]

$$\begin{aligned} R_x &= m^B\left[\ddot{x} \;+\; \ddot{\theta}\,L\,(\mathbf{b}_x \cdot \mathbf{n}_x) \;-\; \dot{\theta}^2\,L\,(\mathbf{b}_y \cdot \mathbf{n}_x)\right] \\ &= m^B\left[\ddot{x} \;+\; \ddot{\theta}\,L\,\cos(\theta) \;-\; \dot{\theta}^2\,L\,\sin(\theta)\right] \end{aligned}$$

[a]Use the rotation table to calculate dot-products.

Dot-multiplication with $\mathbf{n}_y$ gives:[a]

$$\begin{aligned} (R_y - m^B g) &= m^B\left[\ddot{\theta}\,L\,(\mathbf{b}_x \cdot \mathbf{n}_y) \;-\; \dot{\theta}^2\,L\,(\mathbf{b}_y \cdot \mathbf{n}_y)\right] \\ &= m^B\left[\ddot{\theta}\,L\;\text{-}\sin(\theta) \;-\; \dot{\theta}^2\,L\;\cos(\theta)\right] \end{aligned}$$

[a]Use the rotation table to calculate dot-products.

## 11.12 Euler's planar rigid body equation

***Euler's planar rigid body equation*** for a rigid body $B$ in a Newtonian reference frame $N$ is

$$\mathbf{M}_z^{B/B_{cm}} \underset{(10.3)}{=} I_{zz}\;{}^N\boldsymbol{\alpha}^B$$

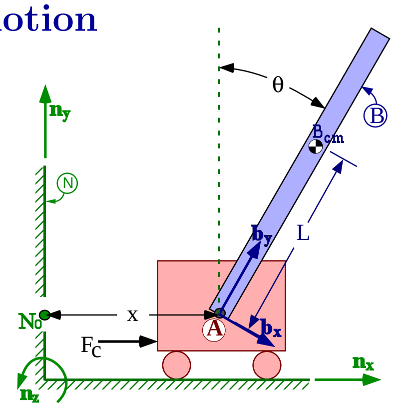
where $\mathbf{M}_z^{B/B_{cm}}$ is the $\mathbf{n}_z$ component of the moment of all forces on $B$ about $B_{cm}$, $I_{zz}$ is the mass moment of inertia of $B$ about the line passing through $B_{cm}$ and parallel to $\mathbf{b}_z$, and ${}^N\boldsymbol{\alpha}^B$ is the angular acceleration of $B$ in $N$. Assembling the terms and subsequent dot-multiplication with $\mathbf{b}_z$ produces

$$L\,\cos(\theta)\,R_x \;-\; L\,\sin(\theta)\,R_y \;=\; \text{-}I_{zz}\,\ddot{\theta}$$

## 11.13 Summary of Newton/Euler equations of motion

Combining the results from Sections 11.11 and 11.12 gives

$$\begin{aligned} F_c \;-\; R_x &= m^A\,\ddot{x} \\ N \;-\; m^A\,g \;-\; R_y &= 0 \\ R_x &= m^B\left[\ddot{x} \;+\; \ddot{\theta}\,L\,\cos(\theta) \;-\; \dot{\theta}^2\,L\,\sin(\theta)\right] \\ (R_y - m^B g) &= m^B\left[\text{-}\ddot{\theta}\,L\,\sin(\theta) \;-\; \dot{\theta}^2\,L\,\cos(\theta)\right] \\ L\,\cos(\theta)\,R_x \;-\; L\,\sin(\theta)\,R_y &= \text{-}I_{zz}\,\ddot{\theta} \end{aligned}$$




The unknown variables[5] in the previous set of equations[6] are $R_x$, $R_y$, $N$, $x$, and $\theta$.

[5]Once $\theta(t)$ is known, $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ are known. Similarly, once $x(t)$ is known, $\dot{x}(t)$ and $\ddot{x}(t)$ are known.

[6]The methods of D'Alembert, Gibbs, Lagrange, and Kane, are more efficient than free-body diagrams for forming equations of motion in that the unknown "constraint forces" $R_x$, $R_y$, and $N$ are automatically eliminated.

## 11.14 Summary of kinematics

This chapter presented a detailed analysis for forming equations of motion for an inverted pendulum on a cart. Although some of the analysis is specific to this problem, many of the kinematic definitions and equations are widely applicable. The important concepts are summarized in the following table.

| Quantity | How determined | Specific to this problem |
|---|---|---|
| Rotation matrix | SohCahToa | $\begin{array}{c\|ccc} {}^{b}R^{n} & \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ \hline \mathbf{b}_x & \cos(\theta) & -\sin(\theta) & 0 \\ \mathbf{b}_y & \sin(\theta) & \cos(\theta) & 0 \\ \mathbf{b}_z & 0 & 0 & 1 \end{array}$ |
| Simple angular velocity | ${}^{N}\boldsymbol{\omega}^{B} \;=\; \pm\, \dot{\theta} \boldsymbol{\lambda}$ | ${}^{N}\boldsymbol{\omega}^{B} \;=\; -\dot{\theta}\, \mathbf{b}_z$ |
| Angular acceleration | ${}^{N}\boldsymbol{\alpha}^{B} \triangleq \frac{{}^{N}d\, {}^{N}\boldsymbol{\omega}^{B}}{dt}$ | ${}^{N}\boldsymbol{\alpha}^{B} \;=\; -\ddot{\theta}\, \mathbf{b}_z$ |
| Position vector | Inspection or vector addition | $\mathbf{r}^{A/N_o} \;=\; x\, \mathbf{n}_x$<br>$\mathbf{r}^{B_{cm}/N_o} \;=\; x\mathbf{n}_x \;+\; L\mathbf{b}_y$ |
| Velocity | ${}^{N}\mathbf{v}^{B_{cm}} \triangleq \frac{{}^{N}d\, \mathbf{r}^{B_{cm}/N_o}}{dt}$ | ${}^{N}\mathbf{v}^{A} \;=\; \dot{x}\, \mathbf{n}_x$<br>${}^{N}\mathbf{v}^{B_{cm}} \;=\; \dot{x}\, \mathbf{n}_x \;+\; L\, \dot{\theta}\, \mathbf{b}_x$ |
| Acceleration | ${}^{N}\mathbf{a}^{B_{cm}} \triangleq \frac{{}^{N}d\, {}^{N}\mathbf{v}^{B_{cm}}}{dt}$ | ${}^{N}\mathbf{a}^{A} \;=\; \ddot{x}\, \mathbf{n}_x$<br>${}^{N}\mathbf{a}^{B_{cm}} \;=\; \ddot{x}\, \mathbf{n}_x \;+\; L\, \ddot{\theta}\, \mathbf{b}_x \;-\; L\, \dot{\theta}^2\, \mathbf{b}_y$ |