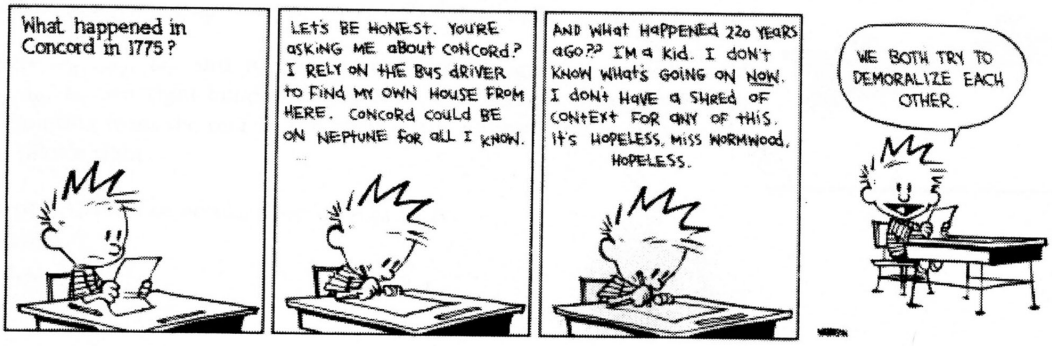
NAME Solutions

Midterm dynamic systems 2008

I certify that I upheld the Stanford Honor code during this exam _____



- Print your name and sign the honor code statement
- You may use your course notes, homework, books, etc.
- Write your answers on this handout
- Where space is provided, show your work to get credit
- If necessary, attach extra pages for scratch work

Problem	Value	Score
1-4	9	×
5	19	
6	9	
7	29	
8	19	×
9	15	
Total	100	

M.1 1 pts. Power-energy principle

The force that best increases the energy of a child on the swing is to push in the direction of $\vec{v}$ when $|\vec{v}|$ is the largest

M.2 2 pts. Review of differentiation

Calculate the ordinary time-derivative of $y = 5^x$.

Result:

$$\frac{dy}{dt} = \frac{d}{dt}(5^x) = \frac{d}{dx}(5^x)\frac{dx}{dt} = \ln 5 \cdot 5^x \cdot \dot{x}$$

M.3 4 pts. Equations and three-dimensional Newtonian Mechanics

$\mathbf{F} = m\mathbf{a}$ is useful for analyzing 3D translational motions of a rigid body. **True/False**

$\mathbf{M} = I\boldsymbol{\alpha}$ is useful for analyzing 3D rotational motions of a rigid body. **True/False**

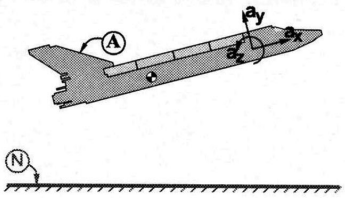
M.4 2 pts. Angular velocity

The definition of angular velocity of $\boldsymbol{\omega} \triangleq \dot{\theta}\,\mathbf{k}$ is a functional operational definition, i.e., in general, it is useful for calculating angular velocity and proving its properties (2D or 3D). **True/False**

M.5 19 pts. Equations of motion for an aircraft.

The motion of an aircraft A in flight over a road N can be characterized by A's angular velocity and the velocity of A_{cm} (A's mass center) as

$$^N\boldsymbol{\omega}^A = \omega_x\,\mathbf{a_x} + \omega_y\,\mathbf{a_y} + \omega_z\,\mathbf{a_z} \qquad {}^N\mathbf{v}^{A_{cm}} = v_x\,\mathbf{a_x} + v_y\,\mathbf{a_y} + v_z\,\mathbf{a_z}$$



where ω_x, ω_y, ω_z and v_x, v_y, v_z are time-dependent variables and $\mathbf{a_x}$, $\mathbf{a_y}$, $\mathbf{a_z}$ are right-handed orthogonal unit vectors fixed in A with $\mathbf{a_x}$ pointing from the rear to the front of the aircraft and $\mathbf{a_z}$ pointing to the pilot's right.

10 pts. Find the acceleration of A_{cm} in N.

Result:

$$^N\mathbf{a}^{A_{cm}} = \dot{v}_x + \omega_y v_z - v_y \omega_z \quad \mathbf{a_x} + \dot{v}_y - \omega_x v_z + \omega_z v_x \quad \mathbf{a_y} + \dot{v}_z + \omega_x v_y - \omega_y v_x \quad \mathbf{a_z}$$

$$^N\vec{a}^{A_{cm}} = \frac{^N d\,{}^N\vec{v}^{A_{cm}}}{dt} = \frac{^A d\,{}^N\vec{v}^{A_{cm}}}{dt} + {}^N\vec{\omega}^A \times {}^N\vec{v}^{A_{cm}}$$

$$\frac{^A d\,{}^N\vec{v}^{A_{cm}}}{dt} = \dot{v}_x\vec{a}_x + \dot{v}_y\vec{a}_y + \dot{v}_z\vec{a}_z$$

$$^N\vec{\omega}^A \times {}^N\vec{v}^{A_{cm}} = \det\begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= (\omega_y v_z - v_y \omega_z)\vec{a}_x - (\omega_x v_z - \omega_z v_x)\vec{a}_y + (\omega_x v_y - \omega_y v_x)\vec{a}_z$$

The resultant of all forces on the aircraft (e.g., forces due to lift, drag, engine thrust, gravity, etc.) is given as

$$\mathbf{F} = F_x\,\mathbf{a_x} + F_y\,\mathbf{a_y} + F_z\,\mathbf{a_z}$$

3 pts. Using Newton's equation $\mathbf{F} = m\,{}^N\mathbf{a}^{A_{cm}}$ (m is the mass of A), form a set of scalar differential equations for $v_x(t)$, $v_y(t)$, and $v_z(t)$.

Result:

$$\dot{v}_x = \frac{F_x}{m} + v_y\omega_z - \omega_y v_z$$

$$\dot{v}_y = \frac{F_y}{m} + \omega_x v_z - \omega_z v_x$$

$$\dot{v}_z = \frac{F_z}{m} + \omega_y v_x - \omega_x v_y$$

$$\vec{F} = m\,{}^N\vec{a}^{A_{cm}}$$

$$F_x\vec{a}_x + F_y\vec{a}_y + F_z\vec{a}_z = m\Big(\big(\dot{v}_x + \omega_y v_z - v_y\omega_z\big)\vec{a}_x + \big(\dot{v}_y - \omega_x v_z + \omega_z v_x\big)\vec{a}_y + \big(\dot{v}_z + \omega_x v_y - \omega_y v_x\big)\vec{a}_z\Big)$$

$\vec{a}_x$: $F_x = m\left(\dot{v}_x + \omega_y v_z - v_y\omega_z\right)$

$\vec{a}_y$: $F_y = m\left(\dot{v}_y - \omega_x v_z + \omega_z v_x\right)$

$\vec{a}_z$: $F_z = m\left(\dot{v}_z + \omega_x v_y - \omega_y v_x\right)$

4 pts. Classify the set of equations governing v_x, v_y, v_z [assume ω_x, ω_y, ω_z are known functions of time, e.g., $\omega_x = \sin(t)$ $\omega_y = \cos(t)$ $\omega_z = \sin(2\,t)$] by picking the relevant qualifiers from the following list.

Uncoupled	**Linear**	**Homogeneous**	**Constant-coefficient**	**1st-order**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Variable-coefficient**	**2nd-order**	**Differential**

2 pts. Given initial values of v_x, v_y, v_z, a numerical value for m, and the functional form of F_x, F_y, F_z (e.g., $F_x = \sin(t)$), how would you simulate the aircraft's motion, i.e., how would you solve the equations of motion to find $v_x(t)$, $v_y(t)$, $v_z(t)$.

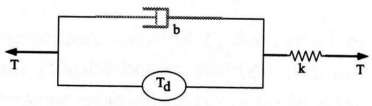
Result: use a computer

M.6 9 pts. Biomechanics of muscle/tendon dynamics

The figure to the right shows a muscle in series with a tendon.

The following identifiers are useful for describing this system.

Quantity	Symbol	Type
Effective spring constant of tendon and muscle	k	constant
Effective damping constant of tendon and muscle	b	constant
Desired tension in muscle	T_d	constant
Tension in muscle	T	variable



The differential equation governing $T(t)$ is

$$b\dot{T} + kT = T_d$$

M.7 5 pts. Find an analytical expression for muscle tension as a function of time.
Use the fact that the muscle is initially relaxed [i.e., $T(0) = 0$].
Result:

$$T(t) =$$

Guess $\Rightarrow T_h(t) = Ce^{pt}$

$\dot{T}_h(t) = Cpe^{pt}$

$b \cdot Cpe^{pt} + kCe^{pt} = 0$

$Ce^{pt}[bp + k] = 0$

$bp + k = 0$

$p = \frac{-k}{b}$

$T_h(t) = Ce^{-\frac{k}{b}t}$

$T_p = A_0$

$\dot{T}_p = 0$

$b \cdot 0 + k \cdot A_0 = T_d$

$T_p = A_0 = \frac{T_d}{k}$

$T(t) = Ce^{-\frac{k}{b}t} + \frac{T_d}{k}$

$T(0) = C + \frac{T_d}{k} = 0$

$C = \frac{-T_d}{k}$

$T(t) = -\frac{T_d}{k}e^{-\frac{k}{b}t} + \frac{T_d}{k}$

$$\boxed{T(t) = \frac{T_d}{k}\left(1 - e^{-\frac{k}{b}t}\right)}$$

Integration by parts

$b\dot{T} + kT = T_d$

$b\dot{T} = T_d - kT$

$\int \frac{b}{T_d - kT}\,dT = \int dt$

$-\frac{b}{k}\ln(T_d - kT) + C = t + D$

$-\frac{b}{k}\ln(T_d - kT) = t + E$

$\ln(T_d - kT) = \frac{-kt}{b} + F$

$T_d - kT = Ge^{-\frac{k}{b}t}$

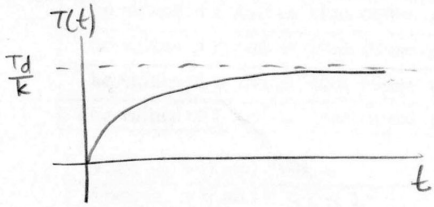
$T = \frac{T_d - Ge^{-\frac{k}{b}t}}{k}$

$T(0) = 0 = \frac{T_d - G}{k}$

$G = T_d$

$$\boxed{T(t) = \frac{T_d}{k}\left(1 - e^{-\frac{k}{b}t}\right)}$$

M.8 2 pts. Sketch the time-history of $T(t)$.
Result:



M.9 2 pts. Find the steady-state and transient portions of $T(t)$.
Result:

$$T(t)_{\text{ss}} = \frac{T_d}{k} \qquad T(t)_{\text{transient}} = -\frac{T_d}{k}e^{-\frac{k}{b}t}$$

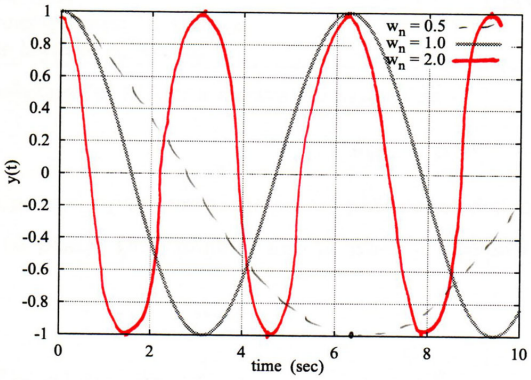
M.10 29 pts. Conceptual understanding of system response versus ω_n

A system's response is governed by the constant-coefficient, linear, second-order, ODE[3]

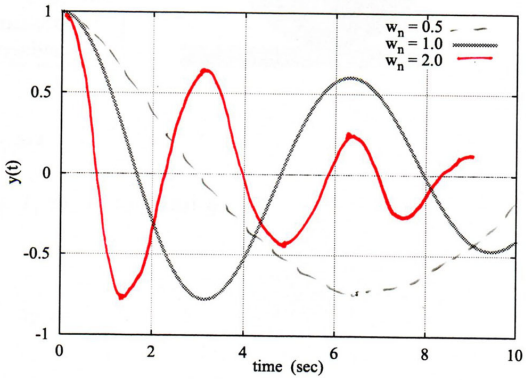
$$\ddot{y} \;+\; 2\zeta\,\omega_n\,\dot{y} \;+\; \omega_n^2\,y \;=\; 0$$

Each curve shown below is a solution $y(t)$ to the ODE for the specified value of ζ, for initial values $y(0)=1$ and $\dot{y}(0)=0$, and for $\omega_n=1$. On each of the four graphs below, **sketch** two more curves of $y(t)$, one for $\omega_n=0.5$ and the other for $\omega_n=2.0$ (use the same value of ζ and same initial values).

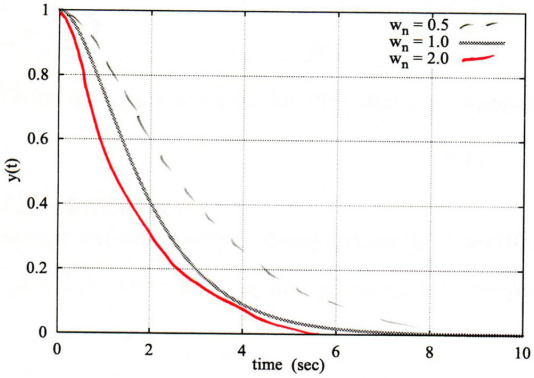
Undamped ($\zeta=0$)



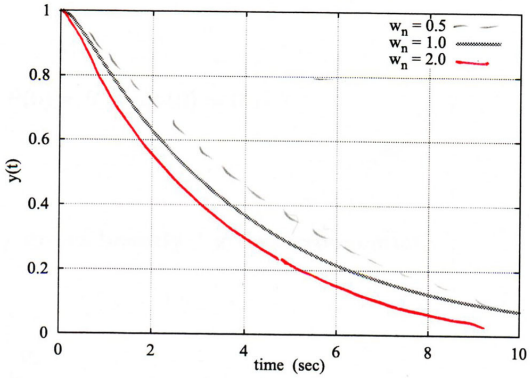
Underdamped (ζ=0.08)



Critically damped ($\zeta=1$)



Overdamped (ζ=2)



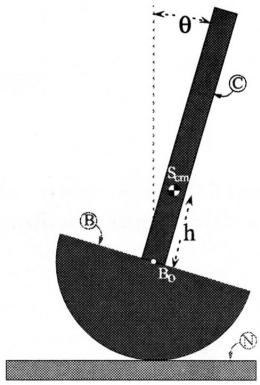
The values of ζ and ω_n that cause $y(t) \to 0$ most quickly are	$\zeta=$ 1	$\omega_n=$ 2
The values of ζ and ω_n that cause $y(t)$ to ***oscillate*** fastest are	$\zeta=$ 0	$\omega_n=$ 2
The values of ζ and ω_n that cause $y(t)$ to ***oscillate*** slowest are	$\zeta=$ .08	$\omega_n=$ .5
The values of ζ and ω_n that cause $y(t)$ to ***decay without oscillation*** slowest are	$\zeta=$ 2	$\omega_n=$ .5

When $\omega_n > 0$ and ζ=0,	ω_d is	negative	zero	positive	real	imaginary	(circle ***all*** that apply)
When $\omega_n > 0$ and $0 < \zeta < 1$,	ω_d is	negative	zero	positive	real	imaginary	
When $\omega_n > 0$ and ζ=1,	ω_d is	negative	zero	positive	real	imaginary	
When $\omega_n > 0$ and $\zeta > 1$,	ω_d is	negative	zero	positive	real	imaginary	
When $\omega_n > 0$ and ζ is imaginary,	ω_d is	negative	zero	positive	real	imaginary	

[3] ω_n and ζ are positive constants called the ***natural frequency*** and ***damping ratio***, respectively.

M.11 19 pts. Solutions to the ODE for a rocking sailboat

The linearized differential equation that governs rolling motions of B on N for the system shown to the right is (you do not have to show this)



$$[I + m\,(h + r_B)^2]\,\ddot{\theta} \;-\; m\,g\,h\,\theta \;=\; 0$$

Quantity	Identifier	Type
Local gravitational constant	g	**positive** constant
Mass of system S consisting of B and C	m	**positive** constant
S's relevant central moment of inertia	I	**positive** constant
Radius of B	r_B	**positive** constant
Distance between B_o and S_{cm}	h	constant
Angle between $\mathbf{n}_y$ and $\mathbf{b}_y$	θ	variable

The solution for $\theta(t)$ can always be written as

$$\theta(t) \;=\; c_1\, e^{p_1\, t} \;+\; c_2\, e^{p_2\, t}$$

$$\ddot{\theta} - \frac{mgh}{I+m(h+r_B)^2}\,\theta = 0 \Longrightarrow \boxed{p^2 = \frac{mgh}{I+m(h+r_B)^2}}$$

where p_1 and p_2 are constants called ***poles*** that depend on I, m, h, r_B, and g.

- **(5 pts.)** Determine p_1 and p_2 when h is ***positive***.

$$p_1 = \sqrt{\frac{mgh}{I+m(h+r_B)^2}} \qquad p_2 = -\sqrt{\frac{mgh}{I+m(h+r_B)^2}}$$

The solution for $\theta(t)$
decays exponentially / decays linearly / oscillates / grows linearly / grows exponentially

- **(7 pts.)** Determine p_1 and p_2 when h is ***zero***.

$$p_1 = \emptyset \qquad p_2 = \emptyset$$

$I + m(r_B)^2)\ddot{\theta} = 0]\,dt$

Write down the solution for $\theta(t)$ that corresponds to $\theta(0) = 0$ and $\dot{\theta}(0) = 0.1$.

$\dot{\theta} = C_1 \Longrightarrow C_1 = 0.1$

$$\theta(t) = 0.1 \cdot t$$

$C_2 = \emptyset$

$\int[\dot{\theta} = C_1]\,dt$

$\boxed{\theta = C_1 t + C_2}$

The solution for $\theta(t)$
decays exponentially / decays linearly / oscillates / grows linearly / grows exponentially

- **(4 pts.)** Determine p_1 and p_2 when h is ***negative***.

$p^2 = \frac{-mg\|h\|}{I+m(h+r_B)^2}$

$$p_1 = i\sqrt{\frac{mg\|h\|}{I+m(h+r_B)^2}} \qquad p_2 = -i\sqrt{\frac{mg\|h\|}{I+m(h+r_B)^2}}$$

The solution for $\theta(t)$
decays exponentially / decays linearly / oscillates / grows linearly / grows exponentially

- **(3 pts.)** Find a range of values of h that produce periodic oscillations of $\theta(t)$ and find the associated period of oscillations.

$h < 0$

$$\tau_{period} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_n} = \boxed{\frac{2\pi\sqrt{I+m(h+r_B)^2}}{\sqrt{mg\|h\|}}}$$

$\zeta = 0$

$$\omega_n^2 = \frac{mgh}{I+m(h+r_B)^2}$$

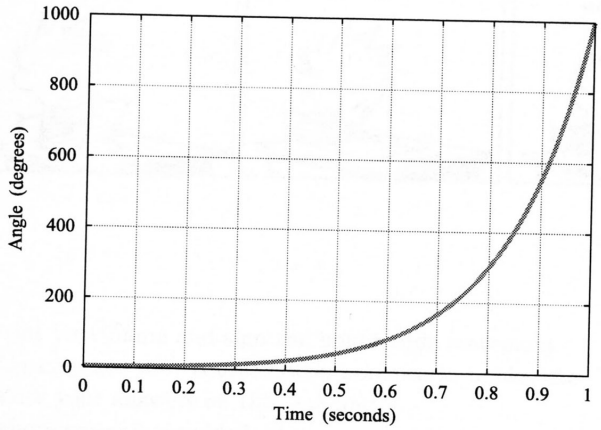
M.12 (15 pts.) System identification for a rocking sailboat

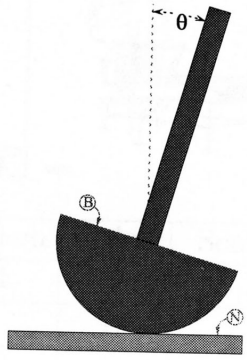
Consider the linear, **constant-coefficient**, homogenous, 2nd-order, ODE

$$\ddot{\theta} - h^2\,\theta = 0$$

where h is a ***positive constant***. Graphed below is the solution $\theta(t)$ when $\theta(0) = 0.2$ rads and $\dot{\theta}(0) = 0$. **Estimate** the numerical value of h to **two or three significant digits** with a simple calculator (using only +, -, *, /, sin, cos, tan, log, and exp).

$h \approx$ _____





Be careful and show your work!

$\ddot{\theta} - h^2\theta = 0$, Find h:

assume $\theta = Ce^{pt}$

$\dot{\theta} = Cpe^{pt}$

$\ddot{\theta} = Cp^2e^{pt}$

Substitute into ODE:

$Cp^2e^{pt} - h^2Ce^{pt} = 0$

$(p^2 - h^2)Ce^{pt} = 0$

$p^2 - h^2 = 0$

$p^2 = h^2$

$p = \pm h \Rightarrow p_1 = h, \; p_2 = -h$

$\theta = C_1e^{p_1t} + C_2e^{p_2t} = C_1e^{ht} + C_2e^{-ht}$

Initial Conditions: $\theta(0) = 0.2, \; \dot{\theta}(0) = 0$

$\theta(0) = 0.2 = C_1 + C_2$

$\dot{\theta} = C_1he^{ht} + -C_2he^{-ht}$

$\dot{\theta}(0) = 0 = C_1h - C_2h = (C_1 - C_2)h$

$C_1 - C_2 = 0 \Rightarrow C_1 = C_2$ plug into

$C_1 = C_2 = 0.1$

$\theta(t) = 0.1e^{ht} + 0.1e^{-ht}$

$\theta(1) = 1000 = 0.1e^{h(1)} + \cancel{0.1e^{-h(1)}}^{\,0}$ (small as h gets big)

(from graph)

$1000 = 0.1e^h$

$10000 = e^h$

$\ln 10000 = \boxed{h = 9.21}$