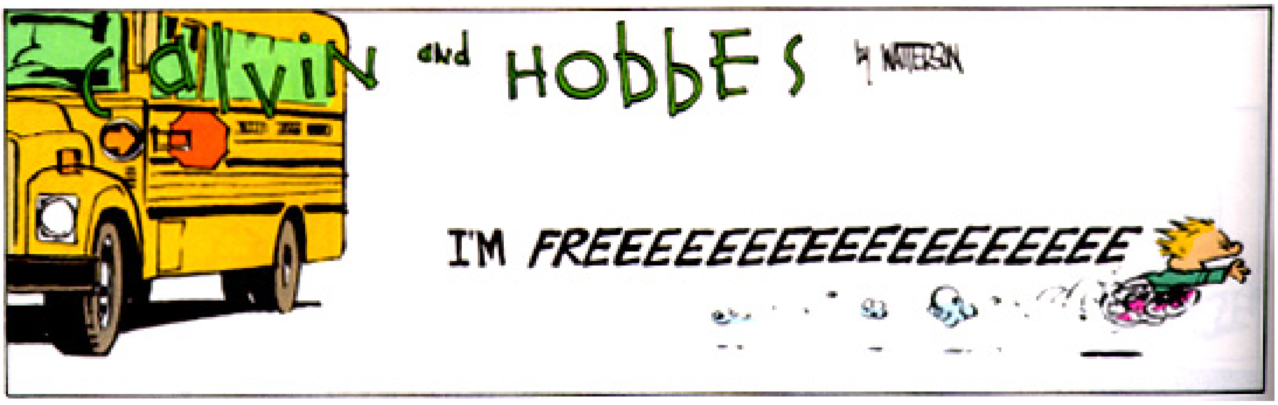
Name ______________________________

ME161 Final. Monday December 12, 2005 3:30-6:30 p.m.

I certify that I upheld the Stanford Honor code during this exam ____________________



- Print your name and sign the honor code statement
- You may use your course notes, homework, books, etc.
- Write your answers on this handout
- Where space is provided, **show your work to get credit**
- If necessary, attach extra pages for scratch work
- Best wishes for a fun vacation. Merry Christmas and Happy New Year!

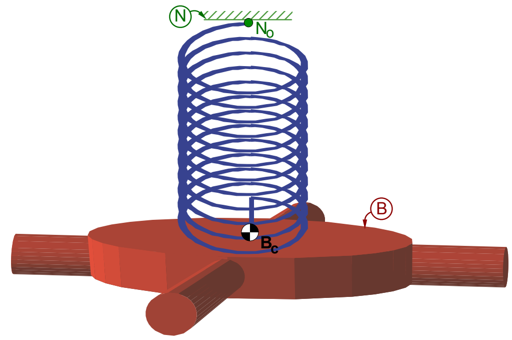
Page	Value	Score
1	2	
2	11	
3	13	
4	7	
5	14	
6	14	
7	10	
8	12	
9	17	
Total	100	

F.1 (2 pts.) Miscellaneous

(a) **(1 pt.)** What is your Meyer-Briggs (human metrics) personality type?
Result:

(b) **(1 pt.)** Suppose you plan on teaching or TAing Dynamic Systems. Provide suggestions for improving the course. Specifically, what you would add, remove, or change in the course (e.g., content, homework, book, lectures, etc.)?
Result:

F.2 (31 pts.) Coupled motions of Wilbur-force pendulum



Shown to the right is a rigid body B that is attached to a spring at B_c, the mass center of B. The other end of the spring is attached to point N_o which is fixed in a Newtonian reference frame N. The equations governing this model of the system are

$$\begin{aligned} m\,\ddot{x} \;+\; k_x\,x \;+\; k_c\,\theta \;&=\; 0 \\ I\,\ddot{\theta} \;+\; k_c\,x \;+\; k_\theta\,\theta \;&=\; 0 \end{aligned}$$

Quantity	Symbol	Type
Mass of B	m	constant
Central moment of inertia of B about vertical	I	constant
Linear spring constant modeling extensional flexibility	k_x	constant
Linear spring constant modeling torsional flexibility	k_θ	constant
Linear spring constant modeling coupled flexibility	k_c	constant
Translational stretch of spring from equilibrium	x	dependent variable
Rotational stretch of spring from equilibrium	θ	dependent variable

(a) **(2 pts.)** Write the ODEs in the matrix form $M\,\ddot{X} + B\,\dot{X} + K\,X = 0$, where $X \triangleq \begin{bmatrix} x \\ \theta \end{bmatrix}$.
Result:

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} k_x & k_c \\ k_c & k_\theta \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) **(9 pts.)** The solution to the previous set of ODEs has the form $X(t) = U * e^{p\,t}$ where p is a constant (to-be-determined) and U is a ***non-zero*** $2{\times}1$ matrix of constants (to-be-determined). Fill in the two blanks in the following polynomial equation that governs the values of $\lambda \triangleq$ -p^2.
Note: the blanks only involve m, I, k_x, k_θ, k_c.
Result:

$$\lambda^2 \;+\; \left(\frac{-k_x}{m} + \frac{-k_\theta}{I}\right) * \lambda \;+\; \frac{k_x\,k_\theta - k_c^2}{m\,I} \;=\; 0$$

(c) **(6 pts.)** For certain values of m, I, k_x, k_θ, and k_c, the matrix $A \triangleq M^{-1} K = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$.

Find the eigenvalues and corresponding eigenvectors of A.

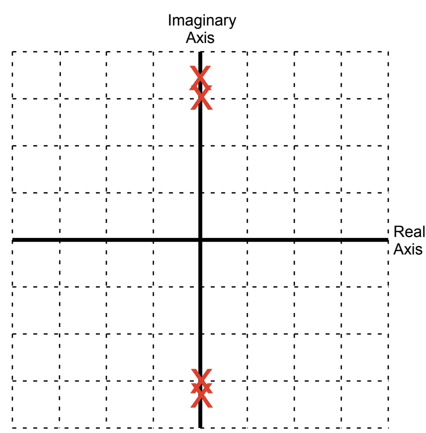
Result:

$$\lambda_1 = 9 \qquad U_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad\qquad \lambda_2 = 11 \qquad U_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(d) **(2 pts.)** Calculate the values of p_1, p_2, p_3, p_4 and draw their locations in the complex plane.

$$p_{1,2} = \pm\sqrt{\text{-}\lambda_1} = \pm\sqrt{\text{-}9} = \pm 3\,i$$

$$p_{3,4} = \pm\sqrt{\text{-}\lambda_2} = \pm\sqrt{\text{-}11} = \pm 3.32\,i$$



(e) **(1 pt.)** The solution is **stable**/**neutrally stable**/**unstable**.

(f) **(4 pts.)** Write the solution for $X(t)$ in terms of the yet-to-be-determined constants c_1, c_2, c_3, c_4, the sine and cosine functions, and t.

Result:

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \left\{ c_1 \sin(3\,t) + c_2 \cos(3\,t) \right\} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left\{ c_3 \sin(3.32\,t) + c_4 \cos(3.32\,t) \right\}$$

(g) **(2 pts.)** Determine c_1, c_2, c_3, c_4 when $x(0) = 0.2$, $\theta(0) = 0$, $\dot{x}(0) = 0$, and $\dot{\theta}(0) = 0$.
Result:

$c_1 = 0$ $\quad c_2 = -0.1$ $\quad c_3 = 0$ $\quad c_4 = 0.1$

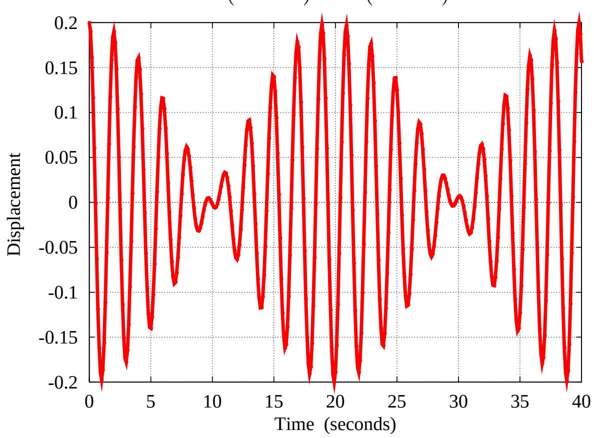
(h) **(2 pts.)** Using the aforementioned initial values, write explicit solutions for $x(t)$ and $\theta(t)$.
Result:

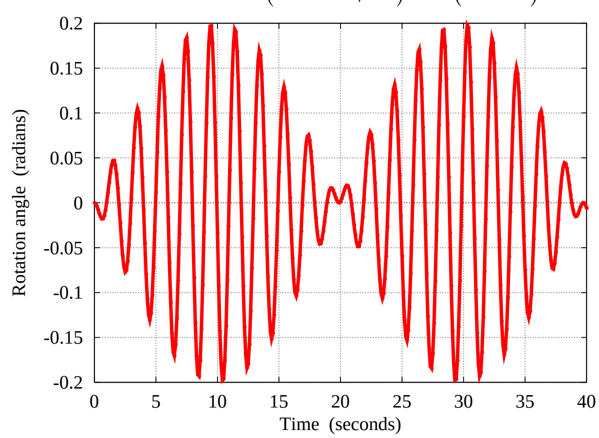
$$x(t) = 0.1\cos(3\,t) + 0.1\cos(3.32\,t)$$
$$\theta(t) = -0.1\cos(3\,t) + 0.1\cos(3.32\,t)$$

(i) **(1 pt.)** Using trigonometric identities, equation (2.19), and $\cos(a) = -\cos(a+\pi)$, show that your previous solution for $x(t)$ and $\theta(t)$ is

$$x(t) = 0.2\sin(-0.16\,t + \frac{\pi}{2})\sin(3.16\,t + \frac{\pi}{2}) = 0.2\cos(0.16\,t)\cos(3.16\,t)$$

$$\theta(t) = 0.2\sin(-0.16\,t)\sin(3.16\,t) = 0.2\sin(0.16\,t + \pi)\sin(3.16\,t)$$





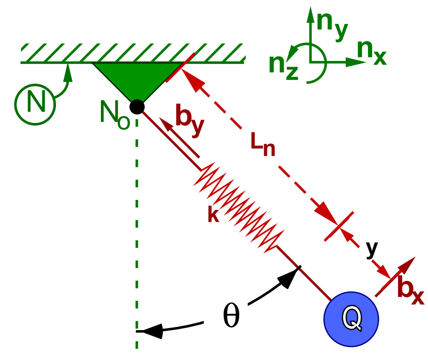
(j) **(2 pts.)** Give an interpretation of the time-behavior of $x(t)$ and $\theta(t)$.
Result:

Both $x(t)$ and $\theta(t)$ exhibit the ***beat phenomena*** with a high-frequency of 3.16 $\frac{\text{rad}}{\text{sec}}$ and a low-frequency of 0.16 $\frac{\text{rad}}{\text{sec}}$.

Since $x(t)$ and $\theta(t)$ are ***coupled***, x's maximum amplitude coincides with θ's minimum amplitude (and vice-versa).

F.3 (38 pts.) Linearization of equations of motion for a swinging spring

A straight, massless, linear spring connects a particle Q to a point N_o which is fixed in a Newtonian reference frame N. The system identifiers and governing equations are shown below.



Quantity	Identifier	Type
Local gravitational constant	g	constant
Mass of Q	m	constant
Natural spring length	L_n	constant
Linear spring constant	k	constant
Spring stretch	y	variable
"Pendulum angle"	θ	variable

$$m\,(L_n + y)\,\ddot{\theta} \;+\; 2\,m\,\dot{\theta}\,\dot{y} \;+\; m\,g\,\sin(\theta) \;=\; 0$$
$$m\,\ddot{y} \;-\; m\,(L_n{+}y)\,\dot{\theta}^2 \;+\; k\,y \;-\; m\,g\,\cos(\theta) \;=\; 0$$

(a) **(1 pt.)** Classify the previous equations by picking the relevant qualifiers from the following list.

Uncoupled	**Linear**	**Homogeneous**	**Constant-coefficient**	**1st-order**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Variable-coefficient**	**2nd-order**	**Differential**

(b) **(3 pts.)** Find the condition on y_{nom} such that $y = y_{\text{nom}}$ **(a constant)** and $\theta = \theta_{\text{nom}} = 0$ satisfy the nonlinear ODEs.

Result:

$$y_{\text{nom}} \;=\; \frac{m\,g}{k}$$

(c) **(10 pts.)** **Using a Taylor series**, linearize the differential equations in perturbations about this nominal solution. In other words, define perturbations of y and θ as

$$\widetilde{y} \triangleq y - y_{\text{nom}} \quad (y_{\text{nom}} \text{ is } \textbf{constant}) \qquad\qquad \widetilde{\theta} \triangleq \theta - \theta_{\text{nom}} \quad (\theta_{\text{nom}} = 0)$$

and form a set of differential equations which are linear in $\widetilde{y}$, $\dot{\widetilde{y}}$, $\ddot{\widetilde{y}}$, $\widetilde{\theta}$, $\dot{\widetilde{\theta}}$, $\ddot{\widetilde{\theta}}$.

Result:

$$m\,(L_n + y_{\text{nom}})\,\ddot{\widetilde{\theta}} \;+\; m\,g\,\widetilde{\theta} \;=\; 0$$
$$m\,\ddot{\widetilde{y}} \;+\; k\,\widetilde{y} \;=\; 0$$

(d) **(7 pts.)** Replace y and θ in the original nonlinear ODEs with

$$y = \widetilde{y} + y_{\text{nom}} \quad (y_{\text{nom}} \text{ is the } \textbf{constant} \text{ determined earlier}) \qquad \theta = \widetilde{\theta} + \theta_{\text{nom}} \quad (\theta_{\text{nom}} = 0)$$

Assuming $\widetilde{y}$, $\dot{\widetilde{y}}$, $\ddot{\widetilde{y}}$ and $\widetilde{\theta}$, $\dot{\widetilde{\theta}}$, and $\ddot{\widetilde{\theta}}$ are small and using the **small-approximations technique** of Chapter 20, form a set of ODEs that are linear in $\widetilde{y}$, $\dot{\widetilde{y}}$, $\ddot{\widetilde{y}}$, $\widetilde{\theta}$, $\dot{\widetilde{\theta}}$, $\ddot{\widetilde{\theta}}$.
Result:

$$m\,(L_n + y_{\text{nom}})\,\ddot{\widetilde{\theta}} \;+\; m\,g\,\widetilde{\theta} \;=\; 0$$

$$m\,\ddot{\widetilde{y}} \;+\; k\,\widetilde{y} \;=\; 0$$

(e) **(1 pt.)** Classify your equations in part (3d) by picking the relevant qualifiers from the following list.

[Uncoupled]	**[Linear]**	**[Homogeneous]**	**[Constant-coefficient]**	**1st-order**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Variable-coefficient**	**[2nd-order]**	**[Differential]**

(f) **(1 pt.)** The linearized ODEs resulting from the Taylor series technique are identical to those obtained from the small-approximations technique. **[True]/False**.

(g) **(5 pts.)** Find analytical solutions for $\widetilde{y}(t)$ and $\widetilde{\theta}(t)$ when $g = 9.8$ m/sec^2, $m = 1$ kg, $L_n = 0.5$ m, $k = 100$ n/m, $\widetilde{y}(0) = 0.1$ m, $\widetilde{\theta}(0) = 1°$, and $\dot{\widetilde{y}}(0) = \dot{\widetilde{\theta}}(0) = 0$

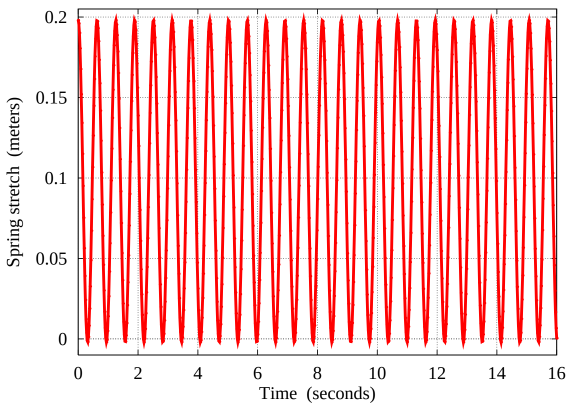
$$\widetilde{y}(t) \;=\; 0.1\,\cos(\sqrt{\tfrac{k}{m}}\,t) \;=\; 0.1\,\cos(10\,t)$$

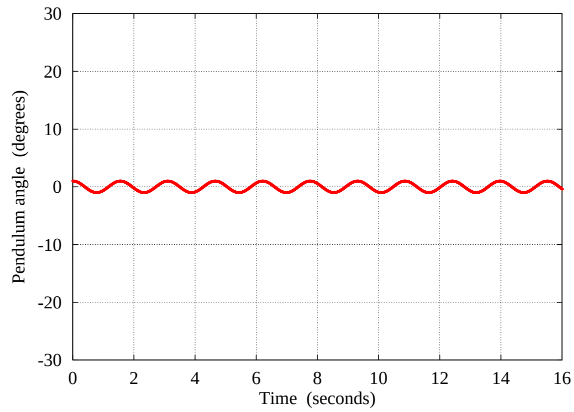
$$\widetilde{\theta}(t) \;=\; 1°\,\cos(\sqrt{\tfrac{g}{L_n + y_{\text{nom}}}}\,t) \;=\; 1°\,\cos(\sqrt{\tfrac{g\,k}{L_n\,k + m\,g}}\,t) \;=\; 1°\,\cos(4.05\,t)$$

(h) **(4 pts.)** Make a **rough** sketch of your linearized solutions for $y(t)$ and $\theta(t)$ for $0 \le t \le 16$ sec, showing the relevant characteristics, e.g., amplitude, frequency, growth, decay, etc.

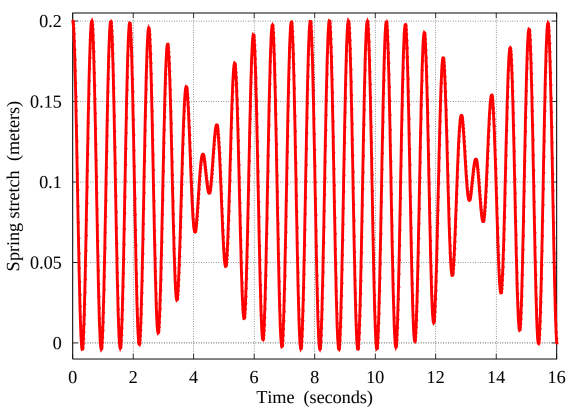
$y(t) \approx \widetilde{y}(t) + y_{\text{nom}} = 0.1\cos(10\,t) + 0.098$

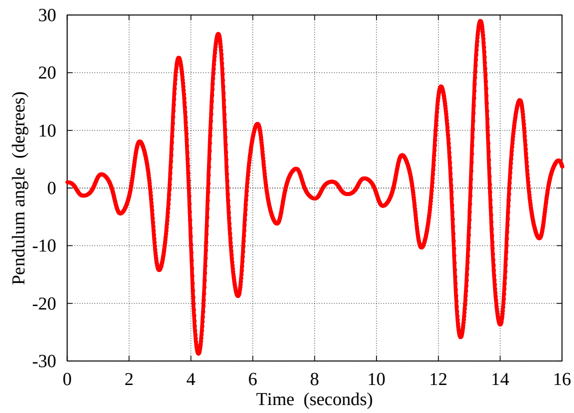
$\theta(t) \approx \widetilde{\theta}(t) + \theta_{\text{nom}} = 1^\circ \cos(4.05\,t)$





(i) **(6 pts.)** The computer solution for $y(t)$ and $\theta(t)$ to the original nonlinear ODEs for $0 \le t \le 16$ sec are plotted below. Comment on the similarities and differences between the plots you produced with the ones below. Give a reason for the similarities and differences.





Similarities: Frequency, maximum amplitude

Differences: Beat phenomenon, varying amplitude

Reason for differences: Coupled and nonlinear

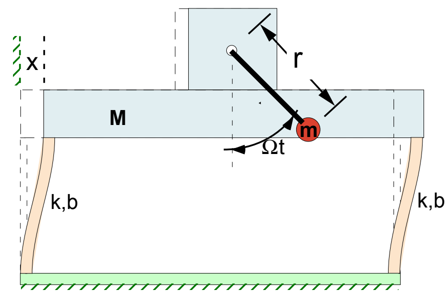
Similarities: Frequency

Differences: Amplitude, beat phenomenon

Reason for differences: Coupled and nonlinear

F.4 (29 pts.) **Dynamic response for an air-conditioner on a building**

An air conditioner is bolted to the roof of a one story building. The air conditioner's motor is unbalanced and its eccentricity is modeled as a particle of mass m attached to the distal end of a rigid rod of length r. When the motor spins with angular speed Ω, it causes the building's roof of mass M to vibrate. The stiffness and material damping in *each* column that supports the roof is modeled as a linear horizontal spring (k) and linear horizontal damper (b). Homework 4.5 showed that the ODE governing the horizontal displacement x of the building's roof is



$$(M+m)\,\ddot{x} \;+\; 2\,b\,\dot{x} \;+\; 2\,k\,x \;=\; m\,r\,\Omega^2\,\sin(\Omega t)$$

(a) **(3 pts.)** Express ω_n, ζ, A in terms of M, m, b, k, r so the ODE is

$$\ddot{x} \;+\; 2\,\zeta\,\omega_n\,\dot{x} \;+\; \omega_n^2\,x \;=\; A\Omega^2\,\sin(\Omega t)$$

Result:

$\omega_n = \sqrt{\frac{2\,k}{M+m}}$ $\qquad \zeta = \frac{b}{\sqrt{(M+m)\,k}}$ $\qquad A = \frac{m\,r}{M+m}$

(b) **(2 pts.)** For certain values of M, m, b, k, and r, this ODE simplifies to

$$\ddot{x} \;+\; 3\,\dot{x} \;+\; 900\,x \;=\; 1\,\text{x}\,10^{-4}\,\Omega^2\,\sin(\Omega t)$$

Calculate numerical values for ω_n and ζ.

Result:

$\omega_n = 30\ \frac{\text{rad}}{\text{sec}}$ $\qquad \zeta = 0.05$

(c) **(7 pts.)** Fill in numerical values in the following expressions for $x_{ss}(t)$, the steady-state part of $x(t)$.

Ω ($\frac{\text{rad}}{\text{sec}}$)	$x_{ss}(t)$
20	$7.94 \text{ x } 10^{-5} * \sin(20\,t + -6.843°)$
30	$0.001 * \sin(30\,t + -90°)$
40	$2.25 \text{ x } 10^{-4} * \sin(40\,t + -170.3°)$

(d) **(14 pts.)** The building's occupants complain that the roof shakes too much. Comment on the effect small variations of M, m, b, k, and r have on: ζ, $\frac{\Omega}{\omega_n}$, and the magnitude of $x_{ss}(t)$. Fill in each element in the table by writing $-$ (decreases), **0** (no effect), + (increases), or ? (if it may decrease *or* increase). For the 2^{nd}-to-last column, assume the air conditioner's normal operating speed is $\Omega = 28\ \frac{\text{rad}}{\text{sec}}$, and for last column, assume $\Omega = 32\ \frac{\text{rad}}{\text{sec}}$.

	ζ	$\frac{\Omega}{\omega_n}$	$\Omega \approx 28\ \frac{\text{rad}}{\text{sec}}$ $\lvert x_{ss}(t)\rvert$	$\Omega \approx 32\ \frac{\text{rad}}{\text{sec}}$ $\lvert x_{ss}(t)\rvert$
Balancing the motor ($r \to 0$)	**0**	**0**	$-$	$-$
Increasing the motor speed Ω	**0**	+	+	$-$
Decreasing the motor speed Ω	**0**	$-$	$-$	+
Adding mass to the roof (increasing M)	$-$	+	+?	?
Removing mass from the roof	+	$-$	$-$?	?
Stiffening the support columns (increasing k)	$-$	$-$	$-$	+
Adding damping to the columns (increasing b)	+	**0**	$-$	$-$

(e) **(3 pts.)** List three ways to change the **motor** and minimize the roof shaking.

Balance the motor, damp the motor, or operate $\Omega \gg \omega_n$ or $\Omega \ll \omega_n$.