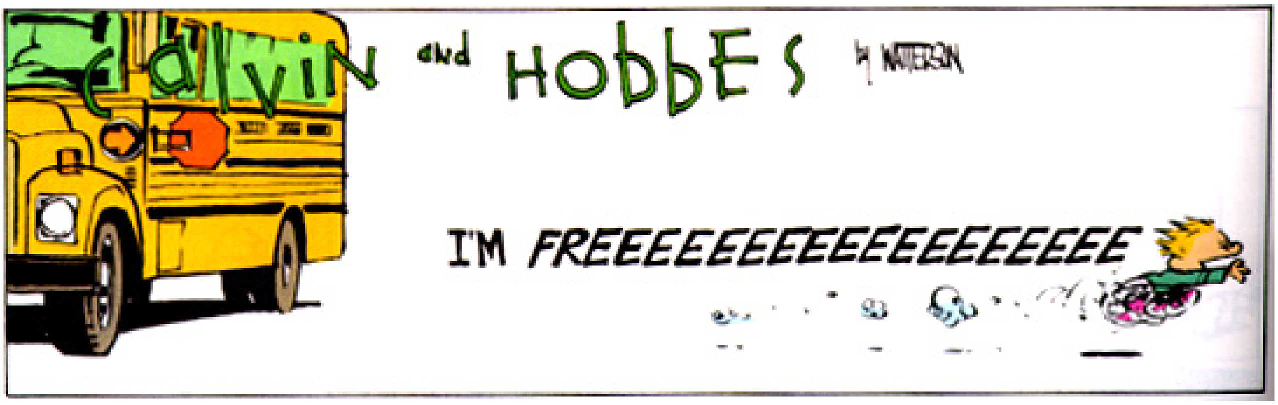
Name ______________________________

ME161 Final. Wednesday December 8, 2004 7:00-10:00 p.m.

I certify that I upheld the Stanford Honor code during this exam ________________



- Print your name and sign the honor code statement
- You may use your course notes, homework, books, etc.
- Write your answers on this handout
- Where space is provided, **show your work to get credit**
- If necessary, attach extra pages for scratch work
- Best wishes for a fun vacation. Merry Christmas and Happy New Year!

Problem	Value	Score
1	6	
2	13	
3	31	
4	20	
5	6	
6	15	
7	9	
Total	100	

Final.1 (6 pts.) Miscellaneous

(a) **(1 pt.)** **Describe one or two things you enjoyed learning in ME161.**
Result:

(b) **(1 pt.)** **Name the engineer, scientist, or mathematician who first did the following:**

Leonard Euler

- Considered sine and cosine as functions (not just the ratio of two sides of a triangle)
- Invented the symbols $\pi = 3.1415...$, $e = 2.71...$, $i = \sqrt{-1}$, $\sum$ for summations, Δx for finite differences, and $f(t)$ for functions
- Discovered a general equation for rotational motions of a rigid body
- Created the foundations of the theory of differential equations
- Discovered a simple technique for numerically integrating differential equations
- Investigated analytical functions of a complex variable
- Invented the Taylor series expansion of a function (concurrent with MacLaurin)
- Studied mathematics under Johann Bernoulli at 14 years old
- Extended Bernoulli's equation for incompressible fluid flow
- Optimized the arrangement of masts on a ship
- "Wrote the book" on the mathematical theory of music
- Worked on cartography, magnetism, fire engines, ship building, and insurance
- Lost 8 of his 13 children in infancy, went half blind at 28 and mostly blind at 59, lost his home to a fire, and still had a positive attitude

(c) **Any real, imaginary, or complex number can be expressed in magnitude-phase form.**

- **(1 pt.)** Express -2 in magnitude-phase form.

$$-2 \;=\; 2\, e^{(\pi + 2n\pi)\, i} \qquad n{=}0, 1, 2, \ldots$$

- **(1 pt.)** **Why does multiplying two negative numbers produce a positive number?** Using magnitude-phase form, show $-2 * -2 = +4$.

$$\left[2\, e^{(\pi+2n\pi)\, i}\right] * \left[2\, e^{(\pi+2n\pi)\, i}\right] \;=\; 4\, e^{(2\pi+4n\pi)\, i} \;=\; \cos(2\pi+4n\pi) + i\, \sin(2\pi+4n\pi) \;=\; 4$$

- **(1 pt.)** **Complex numbers and exponentiation**
  Find ***all*** complex numbers (***in Cartesian form***) equal to the following.

$$\sqrt{i} \;=\; \left[e^{(\frac{\pi}{2}+2n\pi)\, i}\right]^{0.5} \;=\; e^{(\frac{\pi}{4}+n\,\pi)\, i} \;=\; \cos(\tfrac{\pi}{4}+n\pi) + i\, \sin(\tfrac{\pi}{4}+n\pi) \qquad n{=}0, 1, 2, \ldots$$

$$= \; 0.707 + 0.707\, i \qquad \text{or} \qquad -0.707 + -0.707\, i$$

- **(1 pt.)** **Complex numbers and exponentiation**
  Find ***all*** complex numbers (***in Cartesian form***) equal to the following.

$$1^{\frac{1}{2\pi}} \;=\; \left(1\, e^{2n\pi\, i}\right)^{1/(2\pi)} \;=\; e^{n\, i} \;=\; \cos(n) + i\, \sin(n) \qquad n{=}0, 1, 2, \ldots$$

$$= 1, \quad 0.54{+}0.84\, i, \quad -0.42{+}0.91\, i \quad \ldots$$

Final.2 (13 pts.) Sinusoidal transfer function and underdamped vibrations

A dynamic system's response is governed by $\ddot{y} + \dot{y} + y = f(t)$ where $f(t) = 10 \sin(2t)$.

(a) **(2 pts.)** Find the system's transfer function

$$G(s) \triangleq \frac{Y(s)}{F(s)} = \boxed{\frac{1}{s^2 + s + 1}}$$

(b) **(3 pts.)** Is the transfer function stable? **Yes**/**No**.

Explain: $G(s)$ is stable since its poles are in the left-half plane ($-0.5 \pm \sqrt{3}\,i$) .

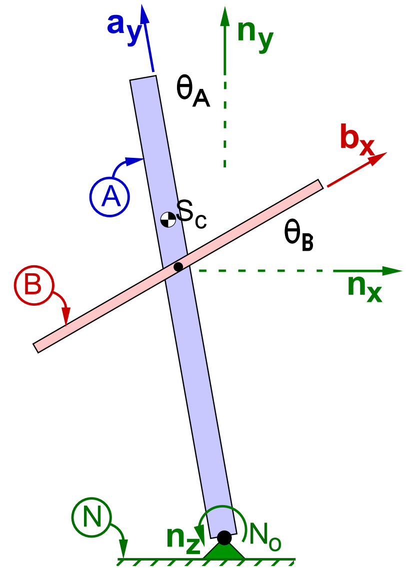
(c) **(5 pts.)** Find numerical values for the magnitude and phase of the sinusoidal transfer function.

Magnitude $= \frac{1}{+\sqrt{13}} = 0.27735$ Phase $=$ `atan2(-2, -3)` $= -2.5536$ rad

(d) **(3 pts.)** Find the steady-state response $y_{ss}(t)$.

$$y_{ss}(t) = 2.7735 * \sin(2t + -2.5536)$$

Final.3 (31 pts.) Solution and stability of a tight-rope walker



A tightrope walker A uses a rigid pole B to balance on a wire at a point N_o that is fixed in a Newtonian reference frame N. Right-handed sets of orthogonal unit vectors are fixed in N, A, and B, with:

- $\mathbf{n}_x$ horizontal and to the right
- $\mathbf{n}_y$ vertically upward
- $\mathbf{a}_y$ directed from N_o to S_c (the mass center of A and B)
- $\mathbf{b}_x$ directed along the balancing pole
- $\mathbf{n}_z = \mathbf{a}_z = \mathbf{b}_z$ perpendicular to the plane in which A and B move

The following identifiers are useful in this analysis.

Quantity	Symbol	Type
Mass of system formed by A and B	m	constant
Distance from N_o to S_c	d	constant
Related to mass distribution of system	I	constant
Central moment of inertia of B for $\mathbf{b}_z$	I^B	constant
Earth's sea-level gravitational constant	g	constant
Feedback-control torque on B from A	T_z	specified variable
Angle between $\mathbf{n}_y$ and $\mathbf{a}_y$	θ_A	dependent variable
Angle between $\mathbf{n}_x$ and $\mathbf{b}_x$	θ_B	dependent variable
Time	t	independent variable

(a) **(1 pt.)** The ***nonlinear*** equations of motion for the tightrope walker are

$$I\,\ddot\theta_A \;+\; \text{-}m\,g\,d\,\sin(\theta_A) \;=\; \text{-}T_z$$
$$I^B\,\ddot\theta_B \;=\; T_z$$

Determine M, K, and G so that the ***linearized*** ODEs can be written in the matrix form $M\,\ddot X \;+\; K\,X \;=\; G\,T_z$ where $X \triangleq \begin{bmatrix}\theta_A\\ \theta_B\end{bmatrix}$.

Result:

$$\begin{bmatrix} I & 0 \\ 0 & I^B \end{bmatrix}\begin{bmatrix}\ddot\theta_A\\ \ddot\theta_B\end{bmatrix} + \begin{bmatrix} \text{-}m\,g\,d & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix}\theta_A\\ \theta_B\end{bmatrix} = \begin{bmatrix} \text{-}1 \\ 1 \end{bmatrix} T_z$$

(b) **(3 pts.)** Consider the homogeneous problem $T_z\!=\!0$. Assume a solution $X(t)\!=\!U e^{pt}$ where p is a constant (to-be-determined) and U is a *non-zero* 2×1 matrix of constants (to-be-determined). Substitute this assumed solution into the matrix equation and find the equation that governs p and U. Express your results in terms of $\lambda \triangleq \text{-}p^2$, $A \triangleq M^{\text{-}1}\,K$, and the 2×2 identity matrix I.

Result:

$$(\text{-}\lambda I + A)\;U \;=\; 0$$

(c) **(4 pts.)** Calculate the 2×2 matrix $A \triangleq M^{-1} K$ in terms of m, g, d, I^B and I.
Result:

$$A = \begin{bmatrix} \frac{-m\,g\,d}{I} & 0 \\ 0 & 0 \end{bmatrix}$$

(d) **(4 pts.)** Calculate λ_i $(i = 1, 2)$.
Result:

$$\lambda_1 = 0 \qquad \lambda_2 = \frac{-m\,g\,d}{I}$$

(e) **(2 pts.)** Find p_1, p_2, p_3 and p_4.
Result:

$$p_1 = 0 \qquad p_3 = +\sqrt{\frac{m\,g\,d}{I}}$$

$$p_2 = 0 \qquad p_4 = -\sqrt{\frac{m\,g\,d}{I}}$$

(f) **(2 pts.)** The solution for $X(t) \triangleq \begin{bmatrix} \theta_A(t) \\ \theta_B(t) \end{bmatrix}$ is **stable/neutrally stable/unstable**.

(g) **(2 pts.)** A larger value of m corresponds to a **less/more** stable solution.
A larger value of I corresponds to a **less/more** stable solution.
A negative value of d corresponds to a **stable/neutrally stable/unstable** solution.[40]

[40]There are tightrope toys built with long downward curved balancing poles so the mass center of S is below N_o.

(h) **(6 pts.)** Find U_1 and U_2, the eigenvectors that correspond to λ_1 and λ_2.
Result:

$$U_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad U_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

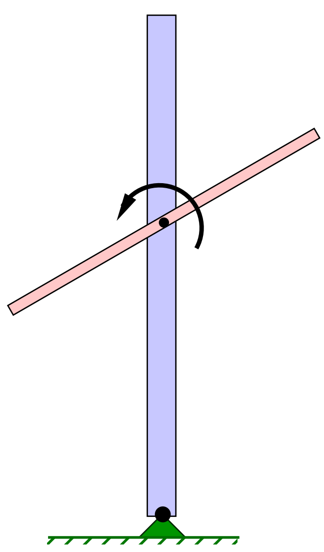
(i) **(2 pts.)** Assemble the solution for $X(t)$[41]
Result:

$$\begin{bmatrix} \theta_A(t) \\ \theta_B(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} t + c_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{+\sqrt{\frac{m\,g\,d}{I}}\,t} + c_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-\sqrt{\frac{m\,g\,d}{I}}\,t}$$

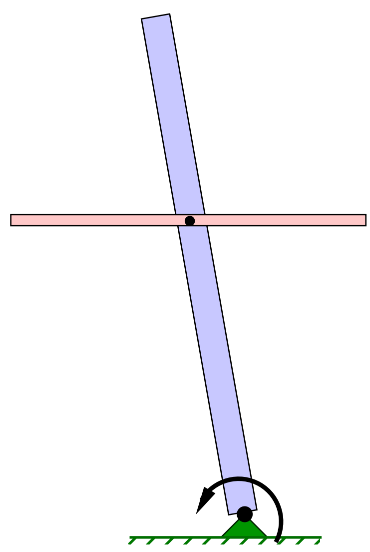
(j) **(1 pt.)** How are the constants c_1, c_2, c_3, c_4 usually determined?

From the initial values of θ_A, θ_B, $\dot{\theta}_A$, and $\dot{\theta}_B$.

(k) **(4 pts.)** Sketch the system moving in its two modes. Clearly show how θ_A and θ_B are changing.



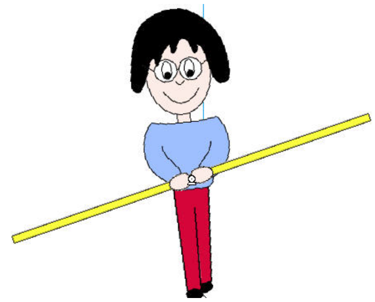
Moving in Mode 1



Moving in Mode 2

[41] When T_z=0, the original set of linear ODEs are uncoupled and one can find the homogeneous solution ***without*** matrices.

Final.4 (20 pts.) State-space feedback control of a tight-rope walker



One way to design an automatic control system for the tight-rope walker is to use the state-space method. The state space-method begins by defining the state matrix Y as

$$Y \triangleq \begin{bmatrix} \theta_A \\ \theta_B \\ \dot{\theta}_A \\ \dot{\theta}_B \end{bmatrix}$$

(a) **(1 pt.)** Suppose the linearized ODEs for the tight-rope walker are

$$\begin{aligned} 100\,\ddot{\theta}_A \;+\; \text{-}500\,\theta_A &= \text{-}T_z \\ 25\,\ddot{\theta}_B &= T_z \end{aligned}$$

Solve for $\ddot{\theta}_A$ and $\ddot{\theta}_B$ in terms of θ_A, θ_B, T_z, etc.,
Result:

$$\begin{aligned} \ddot{\theta}_A &= 5\,\theta_A \;+\; \text{-}0.01\,T_z \\ \ddot{\theta}_B &= 0.04\,T_z \end{aligned}$$

(b) **(4 pts.)** Cast these ODEs into the state-space form $\dot{Y} = A\,Y + B_c\,T_z$.
Result:

$$\begin{bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \\ \ddot{\theta}_A \\ \ddot{\theta}_B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \dot{\theta}_A \\ \dot{\theta}_B \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \text{-}0.01 \\ 0.04 \end{bmatrix} T_z$$

(c) **(4 pts.)** An engineer decides to balance the tight-rope walker using a feedback control law for T_z that is written in terms of the "feedback control constants" k_1, k_2, k_3, k_4, as

$$T_z \;=\; k_1\,\theta_A + k_2\,\theta_B + k_3\,\dot{\theta}_A + k_4\,\dot{\theta}_B \;=\; \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \dot{\theta}_A \\ \dot{\theta}_B \end{bmatrix} \;=\; +K_c\,Y$$

Rewrite the ODEs in the state-space form $\dot{Y} = \overline{A}\,Y$ in terms of k_1, k_2, k_3, k_4, and numbers.
Result:

$$\dot{Y} \;=\; \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 + \text{-}0.01\,k_1 & \text{-}0.01\,k_2 & \text{-}0.01\,k_3 & \text{-}0.01\,k_4 \\ 0.04\,k_1 & 0.04\,k_2 & 0.04\,k_3 & 0.04\,k_4 \end{bmatrix} Y$$

(d) **(1 pt.)** Using physical intuition, ***guess*** at the signs of k_1, k_2, k_3, and k_4 so that if you started with θ_A=10° and θ_B=0°, you could bring this system to rest with θ_A=0° and θ_B=0°. Circle **–** if you think the number is negative, **0** if you think the number is zero, or **+** if you think the number is positive. In other words, should the coefficients of θ_A, θ_B, $\dot{\theta}_A$, and $\dot{\theta}_B$ be negative, zero, or positive in order to bring this system to rest at $\theta_A = \theta_B = 0°$.
Result:

$$T_z \;=\; \underline{(\;+\;0\;-\;)} * \theta_A \;+\; \underline{(\;+\;0\;-\;)} * \theta_B \;+\; \underline{(\;+\;0\;-\;)} * \dot{\theta}_A \;+\; \underline{(\;+\;0\;-\;)} * \dot{\theta}_B$$

(e) **(2 pts.)** The process of solving for $Y(t)$ begins by assuming a solution of the form $Y(t) = U\,e^{\lambda t}$ where λ is a *constant* (to be determined), and U is a *non-zero* 4×1 matrix of constants (to be determined). After substituting this assumed solution into the governing ODE, the matrix equation that governs λ and U is

$$\left(\text{-}\lambda I + \overline{A}\right) U \;=\; 0$$

The unknowns in the previous equation are λ and U. Classify the previous matrix equation by picking the relevant qualifiers from the following list.

Uncoupled	**Linear**	**Homogeneous**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Differential**

(f) **(2 pts.)** How does one solve for λ?

Set the determinant of $\left(\text{-}\lambda I + \overline{A}\right)$ equal to 0

(g) **(2 pts.)** The polynomial equation that relates λ to k_1, k_2, k_3, k_4 is (you do not need to show this)

$$\lambda^4 \;+\; (0.01\,k3 - 0.04\,k4)\,\lambda^3 \;+\; (0.01\,k1 - 0.04\,k2 - 5)\,\lambda^2 \;+\; (0.4\,k4)\,\lambda \;+\; (0.4\,k2) \;=\; 0$$

Determine whether k_i (i=1,2,3,4) must be **negative**, **zero**, **positive**, or **undetermined** for the roots of λ to have negative real parts so that $Y = U e^{\lambda t}$ is stable.
Result:

Feedback Control Constant	Negative, zero, positive, or undetermined
k_1	**negative/zero/positive/undetermined**
k_2	**negative/zero/positive/undetermined**
k_3	**negative/zero/positive/undetermined**
k_4	**negative/zero/positive/undetermined**

(h) **(1 pt.)** Since the feedback control law is $T_z \;=\; k_1\,\theta_A + k_2\,\theta_B + k_3\,\dot{\theta}_A + k_4\,\dot{\theta}_B$ and you know some (or all) of the signs of k_1, k_2, k_3, and k_4, circle the correct signs in the following equation.
Result:

$$T_z \;=\; \underline{(\;+\;0\;-\;)} * \theta_A \;+\; \underline{(\;+\;0\;-\;)} * \theta_B \;+\; \underline{(\;+\;0\;-\;)} * \dot{\theta}_A \;+\; \underline{(\;+\;0\;-\;)} * \dot{\theta}_B$$

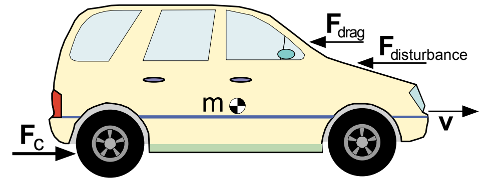
(i) **(2 pts.)** Determine the values of λ that satisfy its polynomial equation when the system is uncontrolled, i.e., $k_1 = k_2 = k_3 = k_4 = 0$.
Result:

$$\lambda_1 \;=\; 0 \qquad\qquad \lambda_2 \;=\; 0$$
$$\lambda_3 \;=\; +\sqrt{5} \qquad\qquad \lambda_4 \;=\; \text{-}\sqrt{5}$$

(j) **(1 pts.)** The uncontrolled system is **stable/neutrally stable/unstable**.

Final.5 **(6 pts.)** **Cruise control for a car**

A model for the speed v of a car of mass m includes unmodeled forces $F_{\text{disturbance}}$, an air-resistance drag force F_{Drag}, and the control force F_c (exerted by the engine and wheels) that tries to move the car at a desired (nominal) speed v_{nom}.



$$m\,\dot{v} \;=\; F_c \;-\; F_{\text{Drag}} \;-\; F_{\text{disturbance}}$$

(a) **(2 pts.)** Determine $F_{c\text{nom}}$, the value of F_c required so $v = v_{\text{nom}}(t)$ when $F_{\text{disturbance}} = 0$. Note: v_{nom} is not necessarily constant.

$$F_{c\text{nom}} \;=\; m\,\dot{v}_{\text{nom}} \;+\; F_{\text{Drag}}$$

(b) **(2 pts.)**

After separating F_c into two terms as shown to the right, rewrite the ODE in terms of $\widetilde{v}$ [the ***error*** between the actual value of v and the desired (nominal) value of v].

$$F_c \;=\; F_{c\text{nom}} \;+\; \widetilde{F}_c \qquad \widetilde{v} \;\triangleq\; v \;-\; v_{\text{nom}}$$

$$m\,\dot{\widetilde{v}} \;=\; \widetilde{F}_c \;+\; F_{\text{disturbance}}$$

(c) **(2 pts.)** What advantage does this model that includes air-resistance (drag) have over a model that does not include air-resistance? Explain.

The advantage of this improved model is that drag is eliminated from the disturbance forces, effectively reducing the magnitude of the disturbance force and improving the control system.

Final.6 **(15 pts.)** **Proportional (P) feedback control for a simple car model**

One choice for $\widetilde{F}_c$ is a ***proportional control law*** of the form $\widetilde{F}_c \;=\; -k_p * \widetilde{v}$ where k_p is a constant.

(a) **(7 pts.)** Assuming $F_{\text{disturbance}}$ is ***constant***, solve for $\widetilde{v}(t)$ in terms of $F_{\text{disturbance}}$, m, k_p, t, and the initial error $\widetilde{v}(0)$.

$$\widetilde{v}(t) \;=\; \widetilde{v}(0)\, e^{\frac{-k_p}{m}\,t} \;+\; \frac{-F_{\text{disturbance}}}{k_p}\left[1 \;-\; e^{\frac{-k_p}{m}\,t}\right]$$

(b) **(1 pt.)** When $F_{\text{disturbance}}$ is constant, the steady-state error is $\widetilde{v}_{\text{ss}} = \frac{-F_{\text{disturbance}}}{k_p}$

(c) **(1 pt.)** Making k_p **small**/**large** & **negative**/**positive** gives the most stable solution for $\widetilde{v}(t)$

(d) **(6 pts.)** Find the values of k_p that satisfy the following specifications for a 1000 kg car.

Design specification	k_p
The steady-state error in response to $F_{\text{disturbance}} = 800$ n is -2 $\frac{\text{m}}{\text{sec}}$	400
With $F_{\text{disturbance}} = 0$, the car goes from rest to 90% of its desired speed within 5 sec	460.5

Final.7 (9 pts.) Proportional-Integral (PI) cruise control for a simple car model

One choice for $\widetilde{F}_c$ is a ***proportional-integral control law*** $\widetilde{F}_c = -k_p * \widetilde{v} + -k_i * \int_{\bar{t}=0}^{t} \widetilde{v}\, d\bar{t}$ where k_p and k_i are constants.

(a) **(3 pts.)** Write a 2^{nd}-order, inhomogeneous, ODE in standard form for $\widetilde{v}(t)$.

$$m\ddot{\widetilde{v}} + k_p\dot{\widetilde{v}} + k_i\widetilde{v} = \dot{F}_{\text{disturbance}}$$

(b) **(1 pt.)** When $F_{\text{disturbance}}$ is constant, the steady-state error is $\widetilde{v}_{\text{ss}} = 0$

(c) **(5 pts.)** Find values of k_p and k_i that satisfy the following specifications for a 1000 kg car.

Design specification	k_p	k_i
With $F_{\text{disturbance}} = 0$, the car goes from rest and settles to within 1% of its desired speed within 5 sec with a maximum overshoot of 2%	$2\,Jblahblah$	$2\,Jblahblah$