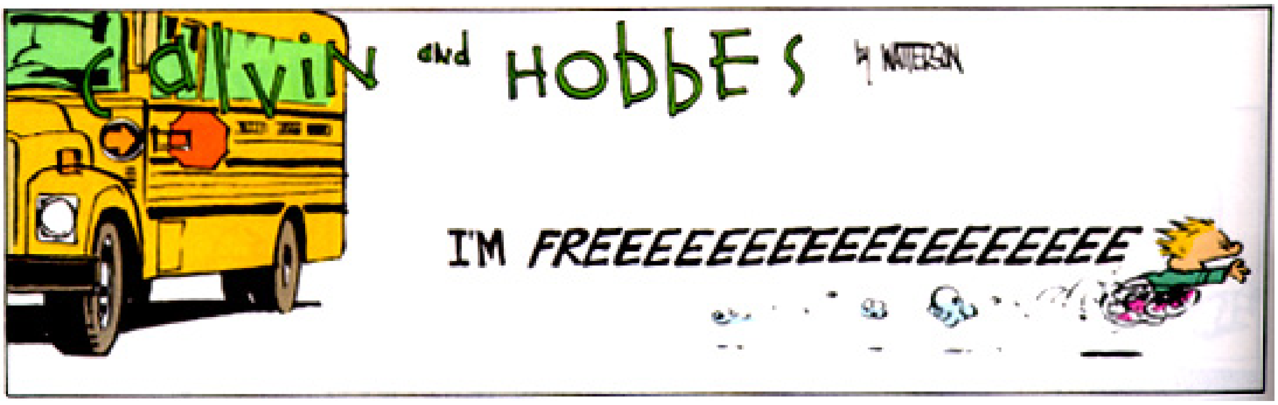
Name ______________________

ME161 Final. Wednesday December 8, 2004 7:00-10:00 p.m.

I certify that I upheld the Stanford Honor code during this exam ______________



- Print your name and sign the honor code statement
- You may use your course notes, homework, books, etc.
- Write your answers on this handout
- Where space is provided, **show your work to get credit**
- If necessary, attach extra pages for scratch work
- Best wishes for a fun vacation. Merry Christmas and Happy New Year!

Problem	Value	Score
1	6	
2	13	
3	31	
4	20	
5	6	
6	15	
7	9	
Total	100	

Final.1 (6 pts.) Miscellaneous

(a) **(1 pt.)** **Describe one or two things you enjoyed learning in ME161.**
Result:

(b) **(1 pt.)** **Name the engineer, scientist, or mathematician who first did the following:**

____

- Considered sine and cosine as functions (not just the ratio of two sides of a triangle)
- Invented the symbols $\pi = 3.1415...$, $e = 2.71...$, $i = \sqrt{-1}$, $\sum$ for summations, Δx for finite differences, and $f(t)$ for functions
- Discovered a general equation for rotational motions of a rigid body
- Created the foundations of the theory of differential equations
- Discovered a simple technique for numerically integrating differential equations
- Investigated analytical functions of a complex variable
- Invented the Taylor series expansion of a function (concurrent with MacLaurin)
- Studied mathematics under Johann Bernoulli at 14 years old
- Extended Bernoulli's equation for incompressible fluid flow
- Optimized the arrangement of masts on a ship
- "Wrote the book" on the mathematical theory of music
- Worked on cartography, magnetism, fire engines, ship building, and insurance
- Lost 8 of his 13 children in infancy, went half blind at 28 and mostly blind at 59, lost his home to a fire, and still had a positive attitude

(c) **Any real, imaginary, or complex number can be expressed in magnitude-phase form.**

- **(1 pt.)** Express -2 in magnitude-phase form.

$$-2 \;=\; \quad e^{(\quad + 2n\pi)\,i} \qquad n{=}0, 1, 2, \ldots$$

- **(1 pt.)** **Why does multiplying two negative numbers produce a positive number?** Using magnitude-phase form, show $-2 * -2 = +4$.

- **(2 pts.)** **Complex numbers and exponentiation**
  Find ***all*** complex numbers (in Cartesian form) equal to the following.

$$\sqrt{i} \;=\; \_\_\_\_\_ + \_\_\_\_\_\, i \qquad \text{or} \qquad \_\_\_\_\_ + \_\_\_\_\_\, i$$

$$1^{\frac{1}{2\pi}} \;=\; 1,$$

Final.2 (13 pts.) Sinusoidal transfer function and underdamped vibrations

A dynamic system's response is governed by $\ddot{y} + \dot{y} + y = f(t)$ where $f(t) = 10\,\sin(2\,t)$.

(a) **(2 pts.)** Find the system's transfer function

$$G(s) \triangleq \frac{Y(s)}{F(s)} =$$

(b) **(3 pts.)** Is the transfer function stable? **Yes/No**.

Explain:

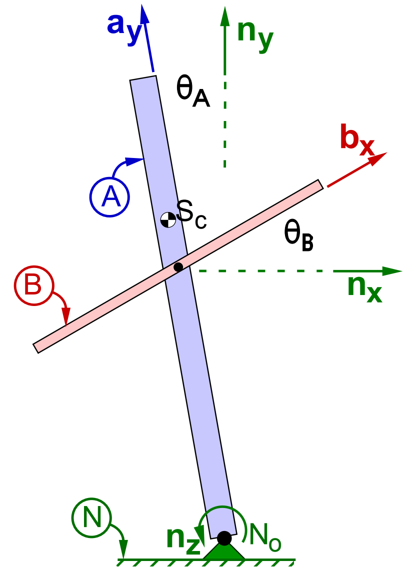
(c) **(5 pts.)** Find numerical values for the magnitude and phase of the sinusoidal transfer function.

Magnitude = Phase = rad

(d) **(3 pts.)** Find the steady-state response $y_{ss}(t)$.

$$y_{ss}(t) =$$

Final.3 (31 pts.) Solution and stability of a tight-rope walker



A tightrope walker A uses a rigid pole B to balance on a wire at a point N_o that is fixed in a Newtonian reference frame N. Right-handed sets of orthogonal unit vectors are fixed in N, A, and B, with:

- $\mathbf{n}_x$ horizontal and to the right
- $\mathbf{n}_y$ vertically upward
- $\mathbf{a}_y$ directed from N_o to S_c (the mass center of A and B)
- $\mathbf{b}_x$ directed along the balancing pole
- $\mathbf{n}_z = \mathbf{a}_z = \mathbf{b}_z$ perpendicular to the plane in which A and B move

The following identifiers are useful in this analysis.

Quantity	Symbol	Type
Mass of system formed by A and B	m	constant
Distance from N_o to S_c	d	constant
Related to mass distribution of system	I	constant
Central moment of inertia of B for $\mathbf{b}_z$	I^B	constant
Earth's sea-level gravitational constant	g	constant
Feedback-control torque on B from A	T_z	specified variable
Angle between $\mathbf{n}_y$ and $\mathbf{a}_y$	θ_A	dependent variable
Angle between $\mathbf{n}_x$ and $\mathbf{b}_x$	θ_B	dependent variable
Time	t	independent variable

(a) **(1 pt.)** The ***nonlinear*** equations of motion for the tightrope walker are

$$I\,\ddot{\theta}_A \;+\; -m\,g\,d\,\sin(\theta_A) \;=\; -T_z$$
$$I^B\,\ddot{\theta}_B \;=\; T_z$$

Determine M, K, and G so that the ***linearized*** ODEs can be written in the matrix form $M\,\ddot{X} \,+\, K\,X \;=\; G\,T_z$ where $X \triangleq \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$.

Result:

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \ddot{\theta}_A \\ \ddot{\theta}_B \end{bmatrix} + \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} T_z$$

(b) **(3 pts.)** Consider the homogeneous problem $T_z=0$. Assume a solution $X(t)=U\,e^{pt}$ where p is a constant (to-be-determined) and U is a *non-zero* 2×1 matrix of constants (to-be-determined). Substitute this assumed solution into the matrix equation and find the equation that governs p and U. Express your results in terms of $\lambda \triangleq -p^2$, $A \triangleq M^{-1}\,K$, and the 2×2 identity matrix I.

Result:

$= \;0$

(c) **(4 pts.)** Calculate the 2×2 matrix $A \triangleq M^{-1} K$ in terms of m, g, d, I^B and I.
Result:

$$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

(d) **(4 pts.)** Calculate λ_i $(i = 1, 2)$.
Result:

$\lambda_1 =$ $\lambda_2 =$

(e) **(2 pts.)** Find p_1, p_2, p_3 and p_4.
Result:

$p_1 =$ $p_3 =$

$p_2 =$ $p_4 =$

(f) **(2 pts.)** The solution for $X(t) \triangleq \begin{bmatrix} \theta_A(t) \\ \theta_B(t) \end{bmatrix}$ is **stable/neutrally stable/unstable**.

(g) **(2 pts.)** A larger value of m corresponds to a **less/more** stable solution.
A larger value of I corresponds to a **less/more** stable solution.

(h) **(6 pts.)** Find U_1 and U_2, the eigenvectors that correspond to λ_1 and λ_2.
Result:

$$U_1 = \begin{bmatrix} \quad \end{bmatrix} \qquad\qquad U_2 = \begin{bmatrix} \quad \end{bmatrix}$$

(i) **(2 pts.)** Assemble the solution for $X(t)$
Result:

$$\begin{bmatrix} \theta_A(t) \\ \theta_B(t) \end{bmatrix} = c_1 \begin{bmatrix} \quad \end{bmatrix} + c_2 \begin{bmatrix} \quad \end{bmatrix} + c_3 \begin{bmatrix} \quad \end{bmatrix} + c_4 \begin{bmatrix} \quad \end{bmatrix}$$

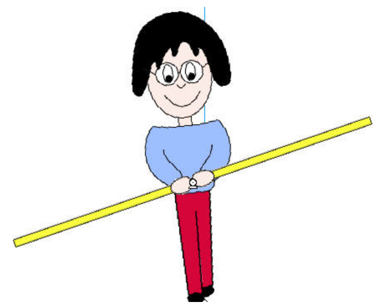
(j) **(1 pt.)** How are the constants c_1, c_2, c_3, c_4 usually determined?

(k) **(4 pts.)** Sketch the system moving in its two modes. Clearly show how θ_A and $\theta_B$are changing.

Moving in Mode 1

Moving in Mode 2

Final.4 (20 pts.) State-space feedback control of a tight-rope walker



One way to design an automatic control system for the tight-rope walker is to use the state-space method. The state space-method begins by defining the state matrix Y as

$$Y \triangleq \begin{bmatrix} \theta_A \\ \theta_B \\ \dot{\theta}_A \\ \dot{\theta}_B \end{bmatrix}$$

(a) **(1 pt.)** Suppose the linearized ODEs for the tight-rope walker are

$$\begin{aligned} 100\,\ddot{\theta}_A \;+\; \text{-}500\,\theta_A &= \text{-}T_z \\ 25\,\ddot{\theta}_B &= T_z \end{aligned}$$

Solve for $\ddot{\theta}_A$ and $\ddot{\theta}_B$ in terms of θ_A, θ_B, T_z, etc.,
Result:

$$\begin{aligned} \ddot{\theta}_A &= \\ \ddot{\theta}_B &= \end{aligned}$$

(b) **(4 pts.)** Cast these ODEs into the state-space form $\dot{Y} = A\,Y + B_c\,T_z$ by completing the following matrices.
Result:

$$\begin{bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \\ \ddot{\theta}_A \\ \ddot{\theta}_B \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \dot{\theta}_A \\ \dot{\theta}_B \end{bmatrix} + \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} T_z$$

(c) **(4 pts.)** An engineer decides to balance the tight-rope walker using a feedback control law for T_z that is written in terms of the "feedback control constants" k_1, k_2, k_3, k_4, as

$$T_z \;=\; k_1\,\theta_A \;+\; k_2\,\theta_B \;+\; k_3\,\dot{\theta}_A \;+\; k_4\,\dot{\theta}_B \;=\; \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \dot{\theta}_A \\ \dot{\theta}_B \end{bmatrix} \;=\; +K_c\,Y$$

Rewrite the ODEs in the state-space form $\dot{Y} \;=\; \overline{A}\,Y$ in terms of k_1, k_2, k_3, k_4, and numbers.
Result:

$$\dot{Y} \;=\; \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} Y$$

(d) **(1 pt.)** Using physical intuition, ***guess*** at the signs of k_1, k_2, k_3, and k_4 so that if you started with θ_A=10° and θ_B=0°, you could bring this system to rest with θ_A=0° and θ_B=0°. Circle **−** if you think the number is negative, **0** if you think the number is zero, or **+** if you think the number is positive. In other words, should the coefficients of θ_A, θ_B, $\dot{\theta}_A$, and $\dot{\theta}_B$ be negative, zero, or positive in order to bring this system to rest at $\theta_A = \theta_B = 0°$.
Result:

$$T_z = (\ + \ 0 \ - \ ) * \theta_A + (\ + \ 0 \ - \ ) * \theta_B + (\ + \ 0 \ - \ ) * \dot{\theta}_A + (\ + \ 0 \ - \ ) * \dot{\theta}_B$$

(e) **(2 pts.)** The process of solving for $Y(t)$ begins by assuming a solution of the form $Y(t) = U\, e^{\lambda t}$ where λ is a *constant* (to be determined), and U is a *non-zero* 4×1 matrix of constants (to be determined). After substituting this assumed solution into the governing ODE, the matrix equation that governs λ and U is

$$\left(\text{-}\lambda I + \overline{A}\right) U = 0$$

The unknowns in the previous equation are λ and U. Classify the previous matrix equation by picking the relevant qualifiers from the following list.

Uncoupled	**Linear**	**Homogeneous**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Differential**

(f) **(2 pts.)** How does one solve for λ?

(g) **(3 pts.)** The polynomial equation that relates λ to k_1, k_2, k_3, and k_4 is (you do not need to show this)

$$\lambda^4 + (0.01\,k3 - 0.04\,k4)\,\lambda^3 + (0.01\,k1 - 0.04\,k2 - 5)\,\lambda^2 + (0.4\,k4)\,\lambda + (0.4\,k2) = 0$$

Determine whether k_i (i=1,2,3,4) must be **negative**, **zero**, **positive**, or **undetermined** for the roots of λ to have negative real parts so that $Y = Ue^{\lambda t}$ is stable.
Result:

Feedback Control Constant	Negative, zero, positive, or undetermined
k_1	**negative/zero/positive/undetermined**
k_2	**negative/zero/positive/undetermined**
k_3	**negative/zero/positive/undetermined**
k_4	**negative/zero/positive/undetermined**

(h) **(1 pt.)** Since the feedback control law is $T_z = k_1\,\theta_A + k_2\,\theta_B + k_3\,\dot{\theta}_A + k_4\,\dot{\theta}_B$ and you know some (or all) of the signs of k_1, k_2, k_3, and k_4, circle the correct signs in the following equation.
Result:

$$T_z = (\ + \ 0 \ - \ ) * \theta_A + (\ + \ 0 \ - \ ) * \theta_B + (\ + \ 0 \ - \ ) * \dot{\theta}_A + (\ + \ 0 \ - \ ) * \dot{\theta}_B$$

(i) **(2 pts.)** Determine the values of λ that satisfy its polynomial equation when the system is uncontrolled, i.e., $k_1 = k_2 = k_3 = k_4 = 0$.
Result:

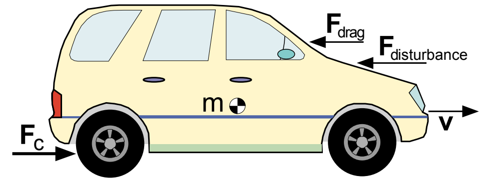
$\lambda_1 =$ $\qquad\qquad$ $\lambda_2 =$

$\lambda_3 =$ $\qquad\qquad$ $\lambda_4 =$

(j) **(1 pts.)** The uncontrolled system is **stable/neutrally stable/unstable**.

Final.5 **(6 pts.)** **Cruise control for a car**

A model for the speed v of a car of mass m includes unmodeled forces $F_{\text{disturbance}}$, an air-resistance drag force F_{Drag}, and the control force F_c (exerted by the engine and wheels) that tries to move the car at a desired (nominal) speed v_{nom}.



$$m\,\dot{v} \;=\; F_c \;-\; F_{\text{Drag}} \;-\; F_{\text{disturbance}}$$

(a) **(2 pts.)** Determine $F_{c\text{nom}}$, the value of F_c required so $v = v_{\text{nom}}(t)$ when $F_{\text{disturbance}} = 0$.

$$F_{c\text{nom}} \;=$$

(b) **(2 pts.)**
After separating F_c into two terms as shown to the right, rewrite the ODE in terms of $\widetilde{v}$ [the ***error*** between the actual value of v and the desired (nominal) value of v].

$$F_c \;=\; F_{c\text{nom}} \;+\; \widetilde{F}_c$$
$$\widetilde{v} \;\triangleq\; v \;-\; v_{\text{nom}}$$

$$=$$

(c) **(2 pts.)** What advantage does this model that includes air-resistance (drag) have over a model that does not include air-resistance? Explain.

Final.6 **(15 pts.)** **Proportional (P) feedback control for a simple car model**

One choice for $\widetilde{F}_c$ is a ***proportional control law*** of the form $\widetilde{F}_c \;=\; \text{-}k_p * \widetilde{v}$ where k_p is a constant.

(a) **(7 pts.)** Assuming $F_{\text{disturbance}}$ is ***constant***, solve for $\widetilde{v}(t)$ in terms of $F_{\text{disturbance}}$, m, k_p, t, and the initial error $\widetilde{v}(0)$.

$$\widetilde{v}(t) \;=$$

(b) **(1 pt.)** When $F_{\text{disturbance}}$ is constant, the steady-state error is $\widetilde{v}_{\text{ss}} =$

(c) **(1 pt.)** Making k_p **small/large** & **negative/positive** gives the most stable solution for $\widetilde{v}(t)$

(d) **(6 pts.)** Find the values of k_p that satisfy the following specifications for a 1000 kg car.

Design specification	k_p
The steady-state error in response to $F_{\text{disturbance}} = 800$ n is -2 $\frac{\text{m}}{\text{sec}}$	
With $F_{\text{disturbance}} = 0$, the car goes from rest to 90% of its desired speed within 5 sec	

Final.7 (9 pts.) Proportional-Integral (PI) cruise control for a simple car model

One choice for $\widetilde{F}_c$ is a ***proportional-integral control law*** $\boxed{\widetilde{F}_c \;=\; \text{-}k_p * \widetilde{v} \;+\; \text{-}k_i * \int_{\bar{t}=0}^{t} \widetilde{v}\, d\bar{t}}$ where k_p and k_i are constants.

(a) **(3 pts.)** Write a 2^{nd}-order, inhomogeneous, ODE in standard form for $\widetilde{v}(t)$.

$=$

(b) **(1 pt.)** When $F_{\text{disturbance}}$ is constant, the steady-state error is $\widetilde{v}_{\text{ss}} =$

(c) **(5 pts.)** Find values of k_p and k_i that satisfy the following specifications for a 1000 kg car.

Design specification	k_p	k_i
With $F_{\text{disturbance}} = 0$, the car goes from rest and settles to within 1% of its desired speed within 5 sec with a maximum overshoot of 2%		